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UNIFORM WEIGHT-FUNCTION APPROXIMATION OF FUNCTIONS WITH VALUES IN HILBERT SPACE

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1. Introduction

In an earlier paper [5] we considered the problem of uniform linear approximation of vector-valued functions using a generalised weight function. This encompasses as particular cases the approximation problem considered by Zuhovickii and Steckin [11], which generalizes the classical Chebyshev approximation theory to functions with values in Hilbert space and a vector-valued analogue of the ordinary relative error approximation for the real case. It was noted that the characterizations of best approximation due to Kolmogorov [6], Zuhovickii [10] and Rivlin and Shapiro [9] for the classical case extend nicely to this case. In the present paper, we continue the study of the problem in [5]. Firstly, a characterization theorem (thm 3.1) for the best weighted approximation in terms of the point evaluation functionals is proved. This generalizes a recent characterization due to Y. Ikebe [2] for the classical Chebyshev problem. In [4] Ivan Singer has noted that the characterization of Y. Ikebe extends to the case of real or complex normed space in terms of the extreme points of the closed unit ball of the dual space. His theorem, however does not seem to directly particularize to the present theorem as the extreme points of the closed unit ball of the function space of vector-valued functions, which we consider, do not admit a representation as plus or minus a point evaluation as in the case of real or complex function spaces. For the case when the approximating functions form a finite dimensional subspace, this theorem reduces to the characterization ([5], thm. 3.1 (3)), which is analogous to the well-known characterization of Zuhovickii (cf. Cheney [1], pp. 73). This leads to the theorem (4.1) which partially generalizes the idea of Chebyshev alternation to this problem. For the unicity of the best approximation, we introduced two conditions in [5], one necessary and the other sufficient. The strong unicity and the continuity of best approximation operator was also established under the assumption of one of these conditions. In this paper, we continue the study of the unicity of best approximation for the case of the functions in several real variables and establish a theorem (thm. 5.1) on the basis of certain differentiability properties. This theorem extends the well-known theorems of Collatz [7] and Rivlin and Shapiro [8] to the present case.

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2. Preliminaries

Let X be a compact Hausdorff space containing at least $n+1$ points and H be a real or complex Hilbert space with an inner product $< >$, a norm $\|\cdot\|$, and a zero element θ. We denote by $\mathcal{C}(X, H)$ the linear space of continuous functions $f: X \to H$, with the usual operations $(f+g)(x)=f(x)+g(x)$, $(\alpha f)(x)=\alpha f(x)$. Let V be a proper subspace of $\mathcal{C}(X, H)$. We call a map $W: X \times H \to H$ a generalized weight map with respect to the subspace V, if it satisfies:

(w_1) W is continuous on $X \times H$

(w_2) W is linear in h i. e. $W(x, \alpha h_1+\beta h_2)=\alpha W(x, h_1)+\beta W(x, h_2)$ for all $\alpha, \beta \in \mathbf{C}$ and $h_1, h_2 \in H$.

(w_3) $p \in V$, $W(x, p(x)) = \theta$ on X implies $p(x) = \theta$ on X.

A non-trivial example of weight map is given in [5]. For $f \in \mathcal{C}(X, H)$, we define $|f| = \sup_{x \in X} \| W(x, f(x)) \|$ and note that $|\cdot|$ is a semi-norm for $\mathcal{C}(X, H)$. For the dual space $\mathcal{C}^*(X, H)$ of $\mathcal{C}(X, H)$ we take the corresponding semi-norm $|L| = \sup_{|g| \leq 1} |Lg|$. Unless otherwise stated we take for $\mathcal{C}(X, H)$ and $\mathcal{C}^*(X, H)$. The topologies of the respective semi-norms $|\cdot|$. We call $p \in V$, a best (weighted) approxi mation to f in V, provided $|f-p| = \inf_{q \in V} |f-q|$.

Let r denote the error function $f-p$ and x^* the point evaluation correspon-ding to an element $x \in X$, the error r and the weight W given by:

$$x^*(g) = \langle W(x, r(x)), \quad W(x, g(x)) \rangle.$$

Let $E_{f,p}$ be the set of all the extremal points of r

$$E_{f,p} = \{x \in X / \| W(x, r(x)) \| = |r|\} \text{ and}$$
$$A = \{x^* |_V / x \in E_{f,p}\}, \text{ where } x^* |_V$$

denotes the restriction of x^* to V.

3. Characterization of best approximation

In the following theorem is given a characterization of the best weighted approximation generalizing the result of Y. Ikebe [2].

Theorem 3.1; *Let V be a subspace of $\mathcal{C}(X, H)$ and $f \in \mathcal{C}(X, H) \sim \overline{V}$. Then $p \in V$ is a best approximation to f in V if and only if the origin o of V^* belongs to the $\sigma(V^*, V)$ — closure of the convex-hull of A.*

Proof: Let us denote by $\Phi(A)$ the $\sigma(V^*, V)$-closure of the convex-hull of A.

Sufficiency: Assume that $o \in \Phi(A)$. We shall construct a linear functional $L \in \mathcal{C}^*(X, H)$, satisfying:

$$(i)\ |L| = 1 \quad (ii)\ L \in V^{\perp} \text{ i. e. } L(V) = 0 \text{ and } (iii)\ |L(f)| = |r|.$$

Then p is a best approximation in V to f. (We note that the results given in [3] p. 182 extend easily to semi-normed spaces, hence conditions (i) — (iii) are both necessary and sufficient for p to be best in V to f). In fact, assuming conditions (i) — (iii) we have for each

$$q \in V,\ |f-q| = |L|\ |f-q| \geqslant |L(f-q)| = |L(f)| = |r| = |f-p|.$$

Next, since $o \in \Phi(A)$, there exists a net L_α in $\Phi(A)$ such that

$$\lim_\alpha L_\alpha = 0.\ \ L_\alpha = \sum c_i (x_i^* |_V),\ x_i^*|_V \in A,\ c_i > 0,\ \sum c_i = 1,$$

the sums ranging over finite number of terms. Consider the corresponding net N_α in $\mathscr{E}^*(X, H)$ defined by:

$$N_\alpha = \sum c_i x_i^*.$$

Since

$$|x_i^*(g)| = |\langle W(x_i, r(x_i)),\ W(x_i, g(x_i))\rangle| \leqslant |r|\ |g|,$$

the net N_α is bounded with respect to the semi-norm $|.|$ by $|r|$. By the $\sigma(\mathscr{E}^*, \mathscr{E})$ compactness of the closed $|r|$ — ball in $\mathscr{E}^*$, (We note that Alaoglu theorem [12], pp. 424, extends easily to semi-normed spaces) the net N_α has a subnet N_{α_β} which converges in $\sigma(\mathscr{E}^*, \mathscr{E})$ topology to an element N of $\mathscr{E}^*$ such that $|N| \leqslant |r|$. Since for each $q \in V$, $N(q) = \lim_\beta N_{\alpha_\beta}(q) = \lim_\beta L_{\alpha_\beta}(q) = 0$, $N(V) = 0$. Also since $N_\alpha(r) = |r|^2$,

$$|N(r)| = \lim N_{\alpha_\beta}(r) = |r|^2.\ \text{ Hence }\ |N| = |r|.$$

Finally set $L = N/|r|$. Then L satisfies *(i)* — *(iii)* proving the sufficiency.

Necessity: Assume that $o \notin \Phi(A)$. Then there exists a $\sigma(V^*, V)$ continuous linear functional L_1 whose real part strongly separates o from $\Phi(A)$. Hence, there exists an element $p_1 \in V$ such that

$$\inf_{x \in X_{f,p}} \{Re\langle W(x, r(x)),\ W(x, p_1(x))\rangle\} > 0$$

This however, contradicts the fact that p is a best approximation in V to f by the extension of Kolmogorov's theorem. (cf. [5] theorem 3.1 (2) (b)), and establishes the necessity.

Taking $H = \mathbf{C}$, the field of complex numbers with the usual inner product $\langle \alpha, \beta\rangle = \alpha\,\bar{\beta}$ and $W(x, y) = y$, the above theorem yields the result of Y. Ikebe [2].

In the particular case when V is finite dimensional with a basis $\{\Phi_1, \ldots \Phi_n\}$, the above theorem also recovers the following corollary ([5], theorem 3.1 (3)):

Corollary 3.2: *If V is finite dimensional, then p is a best approximation to f in V, if and only if, the origin of the n-space $\mathbf{C}^n$ lies in the convex-hull of the set of n-tupples*:

$$Z = \{z = (\langle W(x, r(x)),\ W(x, \varphi_1(x))\rangle,\ \langle W(x, r(x)),\ W(x, \varphi_2(x))\rangle, \ldots$$
$$\ldots \langle W(x, r(x)),\ W(x, \varphi_n(x))\rangle)/x \in X_{f,p}\}.$$

This corollary can be deduced from the above theorem as in [2]. The set $E_{f,p}$ is a compact subset of X, hence the set A is a $\sigma(V^*, V)$ compact subset of V^*. The convex-hull of A is also $\sigma(V^*, V)$ compact, hence $\sigma(V^*, V)$ closed, since V^* is finite dimensional. The assertion of the corollary now easily follows from the isomorphic mapping: $x^* \to (x^*(\varphi_1), \ldots x^*(\varphi_n))$ of V onto the n-space $\mathbf{C}^n$.

4. An alternation type theorem

In [5] we introduced the following condition (T), which was shown to be sufficient for the uniqueness of the best approximation; (T) for $f \in C(X, H)$ given and arbitrary $p, q \in V$, the equation $\langle W(x, r(x)), W(x, q(x))\rangle = 0$ can have at most $n-1$ distinct solutions in X, unless $\langle W(x, r(x)), W(x, q(x))\rangle \equiv 0$ on X. Here $r(x) = f(x) - p(x)$ as before.

Under the assumption of this condition, we obtain the following theorem which partially generalizes the idea of alternation in the classical Chebyshev theory.

Theorem 4.1: *Let H be a real Hilbert space and let f, W and V satisfy the condition (T). Then $p \in V$ is a best approximation to f in V if and only if there exist $n+1$ distinct points $x_i \in X$, such that*

(*i*) $\|W(x_i, r(x_i))\| = |r| = |f-p|$

(*ii*) $\operatorname{sign} \Delta_i = -\operatorname{sign} \Delta_{i+1}$, $i = 1, \dots n$ *where*

$$\Delta_i = \begin{vmatrix} \langle W(x_1\, r(x_1)),\ W(x_1,\ \varphi_1(x_1))\rangle, \dots,\ \langle W(x_{i-1},\ r(x_{i-1})),\ W(x_{i-1}\varphi_1(x_{i-1}))\rangle, \\ \langle W(x_{i+1},\ r(x_{i+1})),\ W(x_{i+1},\ \varphi_1(x_{i+1}))\rangle \dots, \\ \langle W(x_{n+1},\ r(x_{n+1})),\ W(x_{n+1}, \varphi_1(x_{n+1}))\dots, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \langle W(x_1,\ r(x_1)),\ W(x_1,\ \varphi_n(x_1))\rangle, \dots,\ \langle W(x_{i-1},\ r(x_{i-1})),\ W(x_{i-1},\ \varphi_n(x_{i-1}))\rangle, \\ \langle W(x_{i+1},\ r(x_{i+1})),\ W(x_{i+1},\ \varphi_n(x_{i+1}))\rangle \dots, \\ \langle W(x_{n+1},\ r(x_{n+1})),\ W(x_{n+1}\ ,\ \varphi_n(x_{n+1}))\rangle\,., \end{vmatrix}$$

and $\varphi_1, \dots, \varphi_n$ is a basis for V.

Proof: By corollary 3.2, a necessary and sufficient condition for p to be a best approximation to f is that there exist m positive scalars α_i with $\Sigma\, \alpha_i = 1$ and m points $x_i \in X_{f,p}$ such that

$$\sum_{i=1}^{m} \alpha_i \langle W(x_i,\ r(x_i)),\ W(x_i,\ \varphi_j(x_i))\rangle = 0, j = 1,2, \dots, n \qquad (*)$$

By Caratheodory's theorem we may assume that $m \leqslant n+1$. We prove that $m = n+1$, by showing that the assumption $m < n+1$ leads to a cotradiction of condition (T). In fact, by taking $\tilde{x}_i = x_i$, $i = 1, \dots m$; $\{\tilde{x}_i\}_{i=m+1}^{n}$ as a set of distinct points in X different from $\{x_i\}_{i=1}^{n}$ and $\alpha_1 = 0$, $m+1 \leqslant i \leqslant n$, we get

$$\sum_{i=1}^{n} \alpha_1 \langle W(\tilde{x}_i,\ r(\tilde{x}_i)),\ W(\tilde{x}_i,\ \varphi_j(\tilde{x}_i))\rangle = 0,\ j = 1, \dots n.$$

Hence,

$$\det \langle W(\tilde{x}_i,\ r(\tilde{x}_i)),\ W(\tilde{x},\ \Phi_j(\tilde{x}_i))\rangle \,\Big|_{i,j=1}^{n} = 0$$

This contradicts the condition (T). Hence $m = n+1$.

Rewriting (*) as

$$\sum_{i=2}^{n+1} \alpha_i \langle W(x_i, r(x_i)), W(x_i, \Phi_j(x_i))\rangle = -\alpha_1 \langle W(x_1, r(x_1), W(x_1, \Phi_j(x_1))\rangle, \quad (j=1,\dots,n)$$

and solving by the Cramer's rule, we obtain

$$\alpha_i = (-1)^{i-1}\alpha_1 \frac{\Delta_i}{\Delta_1}.$$

Also, we note that $\Delta_i \neq 0$ for all i, since otherwise the condition (T) will be contradicted. This gives

$$\Delta_i \Delta_{i+1} = -\frac{\alpha_i \alpha_{i+1}}{\alpha_1^2} \Delta_1^2 < 0 \qquad (i=1,2,\dots n).$$

Conversely, if

$$\Delta_i \Delta_{i+1} < 0, \qquad (i=1,2,\dots,n)$$

then

$$\alpha_i \alpha_{i+1} > 0, \qquad (i=1,\dots,n).$$

Hence the system (*) has a positive solution and p is a best approximation to f. Theorem 4.1 holds only for the case when $X=[a, b]$ a closed segment.

5. Uniqueness of best approximation

In the case when X is a compact subset of R^k, Rivlin and Shapiro [8] investigated the uniqueness of best approximation in the classical case, on the basis of certain differentiability properties. This generalized the uniqueness result of Collatz [7], which corresponds to the particular case $k=2$. In the following theorem, we generalize the theorem of Rivlin and Shapiro [8] to the present case.

Theorem 5.1: Let X be a compact subset of R^k and $\delta(X)$ be the boundary of X. Let dim $(V)=n$, where $2n \leqslant k+1$ and $\{\Phi_1, \dots, \Phi_n\}$ be a basis of V.

Suppose $f \in \mathcal{C}(X, H)$ and let f, W and V satisfy the following assumptions:
(i) For each $x \in X \sim \delta(X)$, the set

$$S_x = \{W(x, \Phi_j(x))\}_{j=1}^n$$

is an orthogonal subset of H.
(ii) At each point x of $X \sim \delta(X)$, the functions

$$P_i = \langle W(x, f(x)), W(x, \Phi_i(x))\rangle \text{ and } Q_i = \|W(x, \Phi_i(x)\|^2$$

$(i=1,\dots,n)$ possess continuous first partial derivatives

$$P_{ij} = \frac{\partial}{\partial x_j}(P_i), \quad Q_{ij} = \frac{\partial}{\partial x_j}(Q_i), \quad i=1,\dots,n \text{ and } j=1,2,\dots k$$

(*iii*) The matrix

$$\begin{vmatrix} P_1 & P_{11} & \cdots\cdots\cdots & P_{1k} \\ P_2 & P_{21} & \cdots\cdots\cdots & P_{2k} \\ \vdots & \vdots & & \vdots \\ P_n & P_{n1} & \cdots\cdots\cdots & P_{nk} \\ Q_1 & Q_{11} & \cdots\cdots\cdots & Q_{1k} \\ \vdots & \vdots & & \vdots \\ Q_n & Q_{n1} & \cdots\cdots\cdots & Q_{nk} \end{vmatrix}$$

has rank $2n$ at each point of $X \sim \delta(X)$.

(*iv*) f, W and V satisfy the condition (T) on $\delta(X)$.

Under the assumptions (*i*) — (*iv*) there exists a unique best approximation to f in V.

Proof: Let us assume that $p_1 = \sum_{i=1}^{n} \alpha_i \Phi_i$ and $p_2 = \sum_{i=1}^{n} \beta_i \Phi_i$ are two best approximations of f in V. Now if there exists a point $x \in X \sim \delta(X)$, such that $x \in X_{f, p_1}$ as well as $x \in X_{f, p_2}$.

Then for this point

$$\|W(x, f(x) - p_1(x))\| = \|W(x, f(x) - p_2(x))\|,$$

whence

$$2 \operatorname{Re} \sum_{i=1}^{n} (\bar{\alpha}_i - \bar{\beta}_i) \langle W(x, f(x)),\ W(x, \Phi_i(x)) \rangle +$$

$$+ \sum_{i=1}^{n} (|\alpha_i|^2 - |\beta_i|^2) \|W(x, \Phi_i(x))\|^2 = 0$$

i. e.

$$2 \operatorname{Re} \sum_{i=1}^{n} (\bar{\alpha}_i - \bar{\beta}_i) P_i + \sum_{i=1}^{n} (|\alpha_i|^2 - |\beta_i|^2) Q_i = 0,$$

and by the stipulated differentiability properties, we get

$$2 \sum_{i=1}^{n} (\bar{\alpha}_i - \bar{\beta}_i) P_{ij} + \sum_{i=1}^{n} (|\alpha_i|^2 - |\beta_i|^2) Q_{ij} = 0, \qquad j = 1, 2, \ldots. k.$$

From this it follows from the assumption (*iii*) that

$$\alpha_i = \beta_i, \quad i = 1, \ldots, n.$$

On the other hand, if there does not exist a point $x \in X \sim \delta(X)$ such that $x \in X_{f, p_1}$ and $x \in X_{f, p_2}$, then there does not exist a point $x \in X_{f, ((p_1 + p_2)/2)}$ which is contained in $X \sim \delta(X)$. Thus in seeking the best approximation, we may confine the search to $\delta(X)$ only. On $\delta(X)$, the uniqueness follows from the assumption (*iv*) and the theorem 4.1 in [5].

Remarks: The uniqueness assertion of the theorem 5.1, obviously remains valid if the assumptions (*i*), (*ii*) and (*iii*) of the theorem hold at each point of the set $\delta(X)$ (or more generally on any non-empty subset Y of X) and the assumption (*iv*) holds at each point of the set $X \sim \delta(X)$ (more generally on $X \sim Y$).

An example: Let $X = [-1, 1]$, $H = L_2(0, 2\pi)$, $f(x) = 1$.

Take $W(x, y) = y$ and V as the one dimensional subspace spanned by the function $\Phi(x) = (x-1)\cos^2\theta + 1$. Then it is easily verified that for $q = a\Phi$, $|f - q| = \sqrt{\pi(a^2+2)}$. This is minimized for $a = 0$, hence $p = 0$.

$$P_1 = \pi(x+1), \quad Q_1 = \frac{\pi}{4}(3x^2 + 2x + 3),$$

$$P_{11} = \pi \quad \text{and} \quad Q_{11} = \frac{\pi}{4}(6x+2).$$

The matrix

$$\begin{pmatrix} P_1 & P_{11} \\ Q_1 & Q_{11} \end{pmatrix}$$

has rank 2 at each point of $\delta(X)$. The condition (T) is satisfied on $X \sim \delta(X)$.

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