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A NOTE ON ASCOLI'S THEOREM FOR SPACES OF MULTIFUNCTIONS

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1. In [4], Y. F. Lin and D. A. Rose have proved an Ascoli's theorem for a family $\mathcal{F}$ of continuous multifunctions. In the case the family $\mathcal{F}$ consists entirely of single-valued functions this theorem reduces to the usual form of Ascoli's theorem for single-valued functions. The purpose of this note is to show how the single-valued version of Ascoli's theorem implies that for multifunctions ([4]).

Recall first some of the definitions. Let X and Y be two topological spaces. A multifunction $F: X \to Y$ is continuous if and only if for each open set V in Y, the set $F^{-1}(V)$ is open and the set $F^{-1}(Y \setminus V)$ is closed in X. Let the set of all multifunctions on X to Y be denoted by $M(X, Y)$ and let for any $A \subseteq X$, $B \subseteq Y$

$$(A, B) = \{F \in M(X, Y): F(A) \subseteq B\}$$

$$)A, B(= \{F \in M(X, Y): A \subseteq F^{-1}(B)\}.$$

The *compact-open* topology for $M(X, Y)$ is the topology defined by taking the totality of (K, U) and $)L, V($ as subbasic open sets, where K and L are any compact subsets of X and U and V are any open sets in Y. A family $\mathcal{F} \subseteq M(X, Y)$ is *evenly continuous* if and only if for each x in X, each y in Y, and each open neighborhood V of y there exists an open neighborhood U of x and an open neighborhood W of y such that:

(I) if $F \in \mathcal{F}$ and $F(x) \cap W \neq \varnothing$ then $U \subseteq F^{-1}(V)$, and

(II) if, in addition to (I), $F(x) \subseteq V$ then $F(U) \subseteq V$.

When $M(X, Y)$ is replaced by Y^X, two latter definitions coincide with their corresponding single-valued forms [2].

Now the Lin-Rose result reads:

Theorem. *Let X and Y be arbitrary topological spaces and let $M(X, Y; c)$ be the space of all multifunctions on X to Y with the compact-open topology. Then a closed set $\mathcal{F}$ in $M(X, Y; c)$ is compact if, at each point x in X, $\mathcal{F}(x) = \cup \{F(x): F \in \mathcal{F}\}$* has a compact closure in Y, and $\mathcal{F}$ is evenly continuous.

In 2. we just discuss the Ascoli's theorem in the case of compact-open topology. We include the proofs only to make the paper more self-contained and the formulations are given without the separation properties since we do not need them here.

2. For a topological space X; $\mathcal{T}$ stands for the family of all open and $\mathcal{K}$ all compact subsets of X.

2.1. *If* $(X, \mathcal{T}_0)$ *and* $(X, \mathcal{T}_1)$ *are two topological spaces, then* $\mathcal{T}_0 \subseteq \mathcal{T}_1$ *implies* $\mathcal{K}_1 \subseteq \mathcal{K}_0$.

Proof. The identity mapping $i: (X, \mathcal{T}_1) \to (X, \mathcal{T}_0)$ is continuous and if $K \in \mathcal{K}_1$, then $i(K) = K \in \mathcal{K}_0$.

In general, the family $\mathcal{K}_0$ is not contained in $\mathcal{K}_1$, since $\mathcal{K}_0$ may contain such members on which two relative topologies $\mathcal{T}_0$ and $\mathcal{T}_1$ do not coincide. Let us make it more clear.

Let $\mathcal{T}_0$ and $\mathcal{T}_1$ be two topologies on X. Call a subset A of X $(\mathcal{T}_0, \mathcal{T}_1)$-*even* if these two topologies induce the same relative topology on A. For instance, if X is the set of reals, $\mathcal{T}_0$ the usual and $\mathcal{T}_1$ the halfopen interval topology $([.,.))$ then set $A = \left\{\frac{n}{n+1} : n \in N\right\}$ is $(\mathcal{T}_0, \mathcal{T}_1)$-even (and $\mathcal{T}_1$-closed but not $\mathcal{T}_0$-closed),

2.2. *For any two topologies* $\mathcal{T}_0$ *and* $\mathcal{T}_1$ *on* X, *if* A *is* $(\mathcal{T}_0, \mathcal{T}_1)$*-even and* $A \in \mathcal{K}_i$ *then* $A \in \mathcal{K}_{1-i}$, $(i = 0, 1)$.

Proof. If A is $(\mathcal{T}_0, \mathcal{T}_1)$-even, then for $U \in \mathcal{T}_i$, there exists $V \in \mathcal{T}_{1-i}$ such that $A \cap U = A \cap V$. Let $\{V_\varsigma : \zeta \in Z\}$ be a covering of A by the sets from $\mathcal{T}_{1-i}$. Then, since A is $(\mathcal{T}_0, \mathcal{T}_1)$-even for each V_ς, there is a $U_\varsigma \in \mathcal{T}_i$ such that $V_\varsigma \cap A = U_\varsigma \cap A$ and so $\{U_\varsigma : \zeta \in Z\}$ covers A. Since A is $\mathcal{T}_i$-compact there is a finite subcovering $\{U_{\varsigma_i} : i = 1, \ldots, n\}$. Then $\{V_{\varsigma_i} : i = 1, \ldots, n\}$ also covers A and A is $\mathcal{T}_{1-i}$-compact.

Combining 2.1 and 2.2, we also have.

2.3. *If* $\mathcal{T}_0$ *and* $\mathcal{T}_1$ *are two Hausdorff topologies such that* $\mathcal{T}_0 \subseteq \mathcal{T}_1$, *then* $A \in \mathcal{K}_1 \Leftrightarrow A \in \mathcal{K}_0$ *and* A *is* $(\mathcal{T}_0, \mathcal{T}_1)$*-even.*

Recall that a family $\mathcal{F} \subseteq Y^X$ is said to be *evenly continuous* if for each x in X each y in Y, and each (basic) open neighborhood V of y there exist an open neighborhood U of x and a (basic) open neighborhood W of y such that

$$\left.\begin{matrix} f(x) \in W \\ f \in \mathcal{F} \end{matrix}\right\} \Rightarrow f(U) \subseteq V.$$

This condition implies both that all the members of $\mathcal{F}$ are continuous and that $\mathcal{F}$ is even with respect to the point-open and compact-open topology, and the latter being defined by taking for a subbase the sets $\{f : f(K) \subseteq U\}$ where K is compact in X and U belongs to some open base of Y. In the case X is Hausdorff, this topology coincides with what is usually called the compact — open topology on the set Y^X of all continuous functions [1].

2.4. *Let* $\mathcal{F} \subseteq Y^X$ be an evenly continuous family, $\overline{\mathcal{F}}$ the point-open closure of $\mathcal{F}$ and $f_0 \in \overline{\mathcal{F}}$. *If* K *is compact in* X, V *open in* Y *and* $f_0(K) \subseteq V$, *then there exist* $x_1, x_2, \ldots, x_n$ *in* K *and the open neighborboods* $W_i \subseteq V$ *of* $f_0(x_i)$, $i = 1, \ldots, n$ *such that*

$$\left.\begin{matrix} f(x_i) \in W_i \\ f \in \mathcal{F} \end{matrix}\right\} \Rightarrow f(K) \subseteq V, \ i = 1, \ldots, n.$$

Proof. $\mathcal{F}$ being evenly continuous, for each $x \in K$ and $f_0(x)$ in Y and the open neighborhood V of $f_0(x)$, there exist U_x and W_x such that

$$\left.\begin{matrix} f(x) \in W_x \\ f \in \mathcal{F} \end{matrix}\right\} \Rightarrow f(U_x) \subseteq V.$$

Since K is compact there is a finite subcovering $U_{x_1}, U_{x_2}, \ldots, U_{x_n}$ of $\{U_x : x \in K\}$. Put $W_i = W_{x_i} \cap V$. Evidently $f_0(x_i) \in W_i$, $W_i \subseteq V$ and

$$\left.\begin{matrix} f(x_i) \in W_i \\ f \in \mathcal{F} \end{matrix}\right\} \Rightarrow f(U_{x_i}) \subseteq V. \tag{1}$$

Since $f_0 \in \overline{\mathcal{F}}$, the set of $f \in Y^X$ satisfying (1) is not empty. Now, we have for such an f,

$$f(K) \subseteq f(\cup \{U_{x_i} : i = 1, \ldots, n\}) \subseteq V.$$

From 2.4. we conclude the following

2.5. *If* $\mathcal{F} \subseteq Y^X$ *is evenly continuous, then* $\mathcal{F}$ *is even with respect to the point open and compact-open topology on* Y.

2.6. *Let* $\mathcal{F} \subseteq Y^X$ *be evenly continuous. Then,* $\mathcal{F}$ *is closed in the compact--open topology* $\Leftrightarrow$ $\mathcal{F}$ *is closed in the point-open topology.*

Proof. $\Rightarrow$: Let $\mathcal{F}$ be closed in the compact-open topology and $f_0 \in \overline{\mathcal{F}}$ (the point-open closure). Suppose we have a neighborhood of f_0:

$$\mathcal{V} = \{f : f(K_j) \subseteq V_j, \quad j = 1, \ldots, m\}.$$

Then, according to 2.4, we have

$$\mathcal{U}_j = \{f : f(x_i^j) \in W_i^j, \quad i = 1, \ldots, n(j)\}$$

contained in $\{f : f(K_j) \subseteq V_j\}$, and $\mathcal{U}_1 \cap \cdots \cap \mathcal{U}_m \subseteq \mathcal{V}$, so that there exists an $f \in \mathcal{F}$ such that $f \in \mathcal{V}$.

2.7. *(Ascoli's theorem).* $\mathcal{F} \subseteq Y^X$ *is compact in the compact-open topology if*

(I) $\mathcal{F}$ *is closed in compact-open topology*

(II) $\{f(x) : f \in \mathcal{F}\}$ *has a compact closure in* Y *for each point* x *in* X

(III) $\mathcal{F}$ *is evenly continuous.*

Proof. By 2.6, (I) and (III) imply $\mathcal{F}$ is pointwise closed and in addition with (II), it follows that $\mathcal{F}$ is compact in the point-open topology. By 2.5, $\mathcal{F}$ is even, and by 2.2 $\mathcal{F}$ is also compact in the compact-open topology.

3. If $F : X \to Y$ is a multivalued function then we can consider it as a single-valued function from X to 2^Y, where the partitive set of Y is taken with the Vietoris topology [5], and F is continuous (as it has been defined in 1.) if and only if $F : X \to 2^Y$ is continuous [3].

For a subset U of a set Y, let

$$\langle U \rangle = \{A : A \subseteq U\} \quad \text{and} \quad \rangle U \langle = \{A : A \cap U \neq \varnothing, \ A \subseteq Y\}.$$

Then,

3.1. $(K, U) = \{F \in M(X, Y) : F(K) \subseteq \langle U \rangle\}$,

$)K, U(= \{F \in M(X, Y) : F(K) \subseteq \rangle U \langle\}$.

P r o o f. The first relation is evident. For the second, we have

$$F \in)K, U(\Leftrightarrow K \subseteq F^{-1}(U) \Leftrightarrow F(x) \cap U \neq \varnothing,$$

$$\forall x \in K \Leftrightarrow F(x) \in)U\langle, \quad \forall x \in K \Leftrightarrow F(K) \subseteq)U\langle.$$

So the system of sets

$$\{F : F(K) \subset \langle U_0 \rangle \cap)U_1\langle \cap \cdots \cap)U_n\langle\}$$

will be another subbase for $M(X, Y; c)$. The sets $\langle U_0 \rangle \cap)U_1\langle \cap \cdots \cap)U_n\langle$ form the standard basic system for the Vietoris topology on 2^Y.

3.2. *The even continuity of a family* $\mathcal{F} \subseteq M(X, Y)$ *as defined in* [4], *implies the even continuity of the same family* $\mathcal{F} \subseteq (2^Y)^X$.

P r o o f. Take x in X, A in 2^Y and $\langle V_0 \rangle \cap \cdots)V_i\langle \cdots$ to be neighborhood of A. Let $a_i \in A \cap V_i$, $i = 1, 2, \ldots, n$. Then, according to the condition (I) (in 1.), for the pair x, a_i, there exist an open neighborhood W_i of a_i and U_i of x such that

$$\left.\begin{matrix} F(x) \in)W_i\langle \\ F \in \mathcal{F} \end{matrix}\right\} \Rightarrow F(U_i) \subseteq)V_i\langle, \quad i = 1, 2, \ldots, n.$$

Consider $\langle V_0 \rangle \cap \cdots \cap)W_i\langle \cdots$, and let $U = \cap \{U_i\}$. If $F \in \mathcal{F}$ and $F(x) \in \in \langle V_0 \rangle \cap \cdots \cap)W_i\langle \cdots$, then according to the condition (II), $F(U) \subseteq \langle V_0 \rangle \cap \cap \cdots)V_i\langle \cdots$.

So for $x \in X$ and $A \in 2^Y$ and any neighborhood $\langle V_0 \rangle \cap \cdots \cap)V_i\langle \cap \cdots$ of A, U is a neighborhood of x and $\langle V_0 \rangle \cap \cdots \cap)W_i\langle \cap \cdots$ of A such that

$$\left.\begin{matrix} f(x) \in \langle V_0 \rangle \cap \cdots \cap)W_i\langle \cap \cdots \\ f \in \mathcal{F} \end{matrix}\right\} \Rightarrow f(U) \subset \langle V_0 \rangle \cap \cdots \cap)V_i\langle \cdots$$

This proves that $\mathcal{F} \subset (2^Y)^X$ is evenly continuous.

3.3. P r o o f of the Theorem. All three conditions in the Ascoli's theorem (2.7) are satisfied: (I) follows from 3.1, (II) follows from the fact that 2^B, $(B = \overline{\cup \{F(x) : F \in \mathcal{F}\}})$ is compact (see [3] or [5], and the subsets need not be closed), (III) follows from 3.2. Hence, 2.7 implies the conclusion of the Theorem.

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