PUBLICATIONS DE l'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 14 (28) 1972, pp. 97—109

EXPONENTIALLY COMPLETE SPACES III

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(Communicated November 7, 1972)

1. Introduction

Let X be a topological T_1-space and exp (X) the set of all non-empty closed subsets of X, taken with the finite topology. Then, X is exponentially complete if X is homeomorphic to exp (X). This note is a continuation of our two previous notes [3], [4], both being devoted to the study of exponentially complete spaces. Our purpose here is to determine all exp (X) for compact, metric zero-dimensional spaces X, what completes a result of A. Pełczyński from [5]. In particular, we show that there exist exactly nine different topological types of exponentially complete spaces in the class of all compact, metric zero-dimensional spaces.

2. A classification of points of zero-dimensional spaces

The class of all compact, metric zero-dimensional spaces will be denoted by $\mathcal{Z}$. For $X \in \mathcal{Z}$, X_0 denotes the set of all isolated points of X and X_1 the set of those points of X having a (closed and open) neighborhood homeomorphic to the Cantor discontinuum C. In case when X has no isolated point, put $X_0 = \varnothing$ and when X has no point with a neighborhood homeomorphic to C, put $X_1 = \varnothing$. In order to point out the role of X_0 and X_1 in the classification of points of X, call an open and closed subset A of X *topologically minimal* if A is homeomorphic to each of its non-empty, open and closed subset. In the sequel, we will be considering the spaces from $\mathcal{Z}$ only and we will not write the condition $X \in \mathcal{Z}$. Since the term „closed and open" would have appeared so often, it is replaced by the coined and quite customary „clopen".

2.1. *A subset A of X is topologically minimal* $\Leftrightarrow$ *A is an isolated point of X or A is clopen and $A \approx C$.*

P r o o f. $\Rightarrow$: By the definition of a topologically minimal subset, A is clopen and if card $(A) = 1$, A is an isolated point of X. If card $(A) > 1$, then A can have no isolated point. According to the topological caracterisation of C it follows that $A \approx C$.

$\Leftarrow$: Obvious.

2.2. *X_0 and X_1 are open.*

2.3. *$X \setminus \overline{X}_0$ is either empty or $X \setminus \overline{X}_0 = X_1$ and $\overline{X}_1 \approx C$.*

Proof. Suppose $X\setminus\overline{X}_0\neq\varnothing$ Then $x\in X\setminus\overline{X}_0$ has a neighborhood without isolated points and so $x\in X_1$. If $x\in X_1$ then x has a neighborhood having no isolated point and $x\notin\overline{X}_0$. So $x\in X\setminus\overline{X}_0$ and we have $X\setminus\overline{X}_0=X_1$. Since the set $\overline{X}_1$ has no isolated point, $\overline{X}_1\approx C$.

2.4. *The following conditions*

$$\text{(a) } \overline{X\setminus\overline{X}_0}=\varnothing \quad \text{(b) } \overline{X}_0=X \ \textit{and} \ \text{(c) } X_1=\varnothing$$

are equivalent.

Proof. The implications (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c) are obvious. The implication non (a) $\Rightarrow$ non (c) is 2.3.

Let $X_{(0)}=X\setminus(X_0\cup X_1)$, then X is decomposed into the union of X_0, X_1 and $X_{(0)}$. The set $X_{(0)}$ is closed and $X_{(0)}\subseteq X\setminus X_1=\overline{X}_0\cdot X_{(0)}$ need not be a subset of $\overline{X}_1$, and $X_{(0)}$ splits into two mitually disjoint subsets

$$X_{(0)\,(\overline{1})}=X_{(0)}\setminus\overline{X}_1 \text{ and } X_{(0)\,(1)}=X_{(0)}\cap\overline{X}_1.$$

Put $(0)\,(\overline{1})=2$, then X is split into four sets X_0, X_1, X_2, $X_{(0)\,(1)}$. Suppose that $X_0, X_1, \ldots, X_n$ and $X_{(0)}, X_{(0)\,(1)}, \ldots, X_{(0)\,(1)\ldots(n-1)}$ have been defined. Then, put

$$X_{n+1}=X_{(0)\,(1)\ldots(n-1)\,(\overline{n})}=X_{(0)\,(1)\ldots(n-1)}\setminus\overline{X}_n$$

$$X_{(0)\,(1)\ldots(n-1)\,(n)}=X_{(0)\,(1)\ldots(n-1)}\cap\overline{X}_n.$$

Now for each X, the inductive definition of sequences

$$X_0, X_1, X_2, \ldots, X_n, \ldots$$

and

$$X_{(0)}, X_{(0)\,(1)}, \ldots, X_{(0)\,(1)\ldots(n)}, \ldots$$

is complete.

2.5. *For each X and each n,*

$$X=X_0\cup X_1\cup\cdots\cup X_n\cup X_{(0)\,(1)\ldots(n-1)},$$

where the sets X_0, $X_1, \ldots, X_n$, $X_{(0)\,(1)\ldots(n-1)}$ are disjoint $X_{(0)\,(1)\ldots(n-1)}$ is closed and $X_0\cup X_1\cup\cdots\cup X_n$ open.

Proof. The statement 2.5 is true for $n=1$. Suppose 2.5 is true for n. Then, according to the above inductive definition, we have

$$X_{n+1}=X_{(0)\,(1)\ldots(n-1)}\setminus\overline{X}_n\subseteq X_{(0)\,(1)\ldots(n-1)}$$

$$X_{(0)\,(1)\ldots(n-1)\,(n)}=X_{(0)\,(1)\ldots(n-1)}\cap\overline{X}_n\subseteq X_{(0)\,(1)\ldots(n-1)}.$$

The sets X_{n+1} and $X_{(0)\,(1)\ldots(n-1)\,(n)}$ are disjoint and

$$X_{(0)\,(1)\ldots(n-1)}=X_{n+1}\cup X_{(0)\,(1)\ldots(n-1)\,(n)}.$$

By the induction hypothesis, all sets

$$X_0, X_1, \ldots, X_n, X_{n+1}, X_{(0)\,(1)\ldots(n-1)\,(n)},$$

are disjoint and

$$X = X_0 \cup X_1 \cup \cdots \cup X_n \cup X_{n+1} \cup X_{(0)(1)\ldots(n-1)(n)}.$$

The set $X_{(0)(1)\ldots(n-1)(n)}$ is evidently closed when $X_{(0)(1)\ldots(n-1)}$ is closed and then $X_0 \cup X_1 \cup \cdots \cup X_{n+1}$ is open. This proves 2.5.

Let

$$X_\omega = \bigcap \{X_{(0)(1)\ldots(k)} : k = 0, 1, \ldots\}.$$

Then X_ω is a closed subset of X and

$$X = (\bigcup \{X_n : n = 0, 1, \ldots\}) \cup X_\omega.$$

Since $X_\omega \subseteq X_{(0)(1)\ldots(n-1)}$, $X_\omega \cap X_n = \varnothing$ and each point of X belongs to exactly one X_n for $n = 0, 1, \ldots, \omega$. If $x \in X_n$, then x will be called *n-point* of X and n *accumulation order* of x.

For example, if X is a disjoint union of a point x and C, then $X_0 = \{x\}$, $X_1 = C$ and all sets $X_2, X_3, \ldots; X_\omega$ are empty. So x is a 0-point of X and all points in C are 1-points of X. X has no n-point for $n > 1$. If $X = T(C)$, where $T(C)$ is the Cantor discontinuum plus the centers of all removed intervals, then X has 0-points and 2-points and has no n-point for $n = 1, 3, 4, \ldots$

2.6. $\overline{X}_n = X_n \cup (\bigcup \{X_k : k = n+2, n+3, \ldots; \omega\})$.

Proof. From the definition of the sets X_n and $X_{(0)(1)\ldots(n-1)}$ and from 2.5, it follows that

$$\overline{X}_n \supseteq X_{(0)(1)\ldots(n)} = X_{n+2} \cup X_{n+3} \cup \cdots \cup X_\omega. \tag{1}$$

and

$$\overline{X}_n \cap X_{n+1} = \varnothing.$$

Since $X_0 \cup X_1 \cup \cdots \cup X_{n-1}$ is open,

$$\overline{X}_n \subseteq X \setminus (X_0 \cup X_1 \cup \cdots \cup X_{n-1}) \cup X_{n+1} = X_n \cup X_{n+2} \cup \cdots \cup X_\omega. \tag{2}$$

and since $\overline{X}_n \supseteq X_n$, combining (1) and (2) we get the relation in 2.6.

2.7. *If* $X_n = \varnothing$, *then* $X_t = \varnothing$ *for* $t = n+2, \ldots; \omega$.

Proof. If $X_n = \varnothing$, then

$$X_{(0)(1)\ldots(n)} = X_{(0)(1)\ldots(n-1)} \cap \overline{X}_n = \varnothing$$

and

$$X_{n+2} = X_{(0)(1)\ldots(n)} \setminus \overline{X}_{n+1} = \varnothing.$$

2.8. *Let* $X + Y$ *be the topological sum of two spaces* X *and* Y, *then*

$$(X+Y)_n = X_n + Y_n.$$

Proof. For $n = 0$, we have

$$(X+Y)_0 = X_0 + Y_0,$$

since X and Y are clopen subsets of $X+Y$. Suppose that for $n = k$, $(X+Y)_k = X_k + Y_k$. Then, we have to show that

$$(X+Y)_{n+1} = X_{n+1} + Y_{n+1}.$$

Indeed,

$$x\in(X+Y)_{n+1} \Leftrightarrow \begin{cases} x\notin(\overline{X+Y})_n=\overline{X}_n+\overline{Y}_n \\ x\notin(X+Y)_t=X_t+Y_t,\ t=0,\ \dots,\ n-1 \end{cases}$$

$$\Leftrightarrow \begin{cases} x\notin\overline{X}_n \text{ and } x\notin X_t, \text{ for } x\in X \\ \text{or} \\ x\notin\overline{Y}_n \text{ and } x\notin Y_t, \text{ for } x\in Y \end{cases} \Leftrightarrow \begin{cases} x\in X_{n+1} \\ \text{or} \\ x\in Y_{n+1} \end{cases}.$$

2.9. *For each n, there exists an X such that $X_n\neq\varnothing$.*

Proof. We will construct a sequence of spaces $C_0, C_1, \dots, C_n, \dots$ such that $(C_n)_n\neq\varnothing$. Let C_0 be the singleton space, C_1 Cantor discontinuum obtained by the usual process of deleting the open middle third of remaining intervals, C_2 the space $T(C)$ which consists of C_1 plus the centers of all deleted intervals. Evidently, $(C_0)_0\neq\varnothing$, $(C_1)_1\neq\varnothing$ and $(C_2)_2\neq\varnothing$. Let $i_1: C_1\to C_2$ be the inclusion mapping and put $D^k_{-1}=\varnothing$, $C^k_1=C_1\setminus D^k_{-1}$; and if $q_1, q_2, \dots, q_n, \dots$ is the sequence of all isolated points of C_2, let

$$D^k_0=\{q_1, q_2, \dots, q_k\},\quad C^k_2=C_2\setminus D^k_0.$$

It is evident that

$$i_1(C_1)=(C_2)_2=\bigcap_{k=1}^{\infty}\{C^k_2\},$$

and that $(C_2)_2$ is closed in C_2. Before we give the inductive definition of the sequence $C_1, C_2, \dots, C_n, \dots$ we will construct C_3. Denote by S the sequence $\left(\frac{1}{n}\right)$ plus its limit point 0, that is S is the set $\left\{1, \frac{1}{2}, \dots, \frac{1}{n}, \dots, 0\right\}$ with its relative topology of the real line. Let $C_2\times S$ be the topological product of two spaces C_2 and S. Let $i_2: C_2\to C_2\times S$ be given by $i_2(x)=(x, 0)$. Let

$$Q^k_1=i_1(C^k_1)\times\left\{\frac{1}{k}\right\},\quad C_3=(C_2\times\{0\})\cup\left(\bigcup_{k=1}^{\infty}Q^k_1\right),$$

$$D^k_1=\bigcup_{t=1}^{k}\{Q^t_1\},\quad C^k_3=C_3\setminus\left(D^k_1\cup\tilde{D}^k_0\right),\quad \tilde{D}^k_0=i_{12}\left(D^k_0\right).$$

Then, Q^k_1 and C^k_3 are closed and open, and

$$i_{12}(C_1)=(C_3)_3=\bigcap_{k=1}^{\infty}\{C^k_3\},$$

(what will be proved in the induction step from n to $n+1$) and where $i_{12}=i_2\circ i_1$.

Now suppose the sequence $C_1, C_2, C_3, \dots, C_m$ has been defined as well as the sequence of mappings

$$C_1\xrightarrow{i_1}C_2\xrightarrow{i_2}C_3\to\cdots\to C_{m-1}\xrightarrow{i_{m-1}}C_m,\quad i_{rs}=i_s\circ\cdots\circ i_{r+1}\circ i_r.$$

Suppose also that D^k_{t-2} and C^k_t, $t=0, 1, \dots, m-2$ have been defined and that they are open and closed in C_t and that

(1) $$i_{1t-1}(C_1)=(C_t)_t=\bigcap_{k=1}^{\infty}\{C^k_t\},\quad \text{for } t=2, 3, \dots, m,$$

and

$$(C_t)_{t+1}=\varnothing\,,\ (C_t)_{t+p}=\varnothing\,,\ p=1,2,\ldots$$

Now C_{m+1} is constructed as follows. Let $C_m\times S$ has the product topology and let $i_m: C_m\to C_m\times S$ be given by $i_m(x)=(x,0)$. The mapping i_m is an imbedding in a trivial manner. Let

$$Q^k_{m-1}=i_{m-1}(C^k_{m-1})\times\frac{1}{k}\,,\ C_{m+1}=(C_m\times\{0\})\cup\left(\bigcup_{t=1}^{\infty}\{Q^t_{m-1}\}\right),$$

$$D^k_{m-1}=\left(\bigcup_{t=1}^{k}\{Q^t_{m-1}\}\right)\cup\left(\bigcup_{t=1}^{k}\{D^t_{m-2}\}\times\{0\}\right),\ C^k_{m+1}=C_{m+1}\setminus D^k_{m-1}\,.$$

Since C^k_{m-1} and D^k_{m-2} are clopen in C_{m-1} and C_m respectively, Q^k_{m-1}, D^k_{m-1} and C^k_{m+1} will be clopen in C_{m+1}. According to (1),

$$i_{1m}(C_1)=i_m(i_{1m-1}(C_1))=i_m((C_m)_m),$$

and for $x\in C_1$,

$$i_{1m-1}(x)\times\frac{1}{k}\to i_{1m-1}(x)\times\{0\}=i_m(i_{1m-1}(x))=i_{1m}(x),$$

what shows that $i_{1m}(C_1)$ is in the boundary of the open set $\bigcup_{k=1}^{\infty}\{Q^k_{m-1}\}$. Let $(x,0)\in C_m\times\{0\}\setminus i_{1m}(C_1)=C_m\times\{0\}\setminus(C_m)_m\times\{0\}$. Then, $x\notin(C_m)_m$ and there exists a k such that $x\notin C^k_m$. Since $C^k_m\supseteq i_{m-1}(C^k_{m-1})$, $x\notin i_{m-1}(C^k_{m-1})$. If a suquence from $\bigcup_{k=1}^{\infty}\{Q^k_{m-1}\}$ converges to $(x,0)$, then $x\in i_{m-1}(C^k_{m-1})$ for each k. Therefore, no sequence from $\bigcup_{k=1}^{\infty}Q^k_{m-1}$ converges to $(x,0)\in C_m\times\{0\}\setminus i_{1m}(C_1)$, and

$$\partial\left(\bigcup_{k=1}^{\infty}\{Q^k_{m-1}\}\right)=i_{1m}(C_1)\,.$$

Since $i_{1m-1}(C^k_{m-1})\times\left\{\frac{1}{k}\right\}$ is clopen and homeomorphic to C^k_{m-1}, for $x\in C_1$ the point $i_{1m-1}(x)\times\left\{\frac{1}{k}\right\}$ belongs to $(C_{m+1})_{m-1}$ and the point $i_{1m-1}(x)\times\{0\}=i_{1m}(x)$ to $\overline{(C_{m+1})}_{m-1}$. According to 2.6,

$$\overline{(C_{m+1})}_{m-1}=(C_{m+1})_{m-1}\cup(C_{m+1})_{m+1}\cup(C_{m+1})_{m+2}\cup\cdots$$

Now $i_{1m-1}(x)\in(C_m)_m\subseteq\overline{(C_m)}_{m-2}=(C_m)_{m-2}\cup(C_m)_m$ and since $C_m\times\{0\}\setminus\setminus(C_m)_m\times\{0\}$ is open in C_{m+1}, we will have $(C_m)_{m-2}\times\{0\}\subseteq(C_{m+1})_{m-2}$ (every point has a clopen neighborhood contained in $C_m\times\{0\}\setminus(C_m)_m\times\{0\}$). So we also have

$$i_{1m-1}(x)\times 0=i_m(x)\in\overline{(C_{m+1})}_{m-2}=(C_{m+1})_{m-2}\cup(C_{m+1})_m\cup(C_{m+1})_{m+1}\cdots$$

Therefore,

$$i_{1m}(x)\in\overline{(C_{m+1})_{m-1}}\cap\overline{(C_{m+1})_{m-2}}=(C_{m+1})_{m+1}\cup(C_{m+1})_{m+2}\dots \tag{2}$$

The sets Q^k_{m-1} are open, so $Q^k_{m-1}\cap(C_{m+1})_m=\varnothing$, and we see from (2) that $i_{1m}(C_1)\cap(C_{m+1})_m=\varnothing$. Since $C_m\setminus(C_m)_m$ contains no m-point, we also have

$$(C_m\times 0\setminus(C_m)_m\times 0)\cap(C_{m+1})_m=\varnothing,$$

and we have proved that $(C_{m+1})_m$ is empty and so are $(C_{m+1})_{m+2}$, $(C_{m+1})_{m+3},\dots$ Therefore, $(C_{m+1})_m=\varnothing$ and

$$i_{1m}(C_1)=i_m((C_m)_m)=(C_{m+1})_{m+1}.$$

According to the induction hypothesis,

$$(C_m)_m=\bigcap_{k=1}^{\infty}(C_m\setminus D^k_{m-2})$$

and

$$(C_m)_m\times 0=\bigcap_{k=1}^{\infty}(C_m\times 0\setminus D^k_{m-2}\times 0)=C_m\times 0\setminus\bigcup_{k=1}^{\infty}D^k_{m-2}\times 0.$$

So we have

$$\bigcap_{k=1}^{\infty}C^k_{m+1}=\bigcap_{k=1}^{\infty}(C_{m+1}\setminus D^k_{m-1})=C_{m+1}\setminus\bigcup_{k=1}^{\infty}\left(\bigcup_{t=1}^{k}Q^t_{m-1}\right)\cup\left(\bigcup_{t=1}^{k}D^t_{m-2}\times 0\right)$$

$$=C_m\times 0\setminus\left(\bigcup_{k=1}^{\infty}D^k_{m-2}\times 0\right)=(C_m)_m\times 0=i_{1m}(C).$$

Since the induction hypothesis is proved for $m+1$, the existence of the sequence of spaces $C_0, C_1,\dots,C_n,\dots$ such that $(C_n)_n\neq\varnothing$ is established.

2.10. *There exists an X such that $(X)_\omega\neq\varnothing$.*

Proof. By 2.9, there exists a sequence $C_0, C_1,\dots,C_n,\dots$ with $(C_n)_n\neq\varnothing$. Let Σ be the topological sum of these spaces and X one-point compactification of Σ. Then, as it is easy to see, $(X)_\omega\neq\varnothing$.

3. n-points of exp (X). The set $(\exp(X))_n$ of all n-points of $\exp(X)$ will be denoted by $\exp(X)_n$. Now we prove a sequence of relations and the first of them was proved in [5].

3.1. (I) $\exp(X)_0=\langle X_0\rangle$

(II) $\exp(X)_1=\rangle X_1\langle$ [1)]

(III) $\exp(X)_2=\langle X_0\cup X_2;\ X_2\rangle$

(IV) $\exp(X)_3=\langle X_0\cup X_3;\ X_3\rangle$

(V) $\exp(X)_4=\langle X_0\cup X_2\cup X_4;\ X_4\rangle$

(VI) $\exp(X)_5=\langle X_0\cup X_2\cup X_3\cup X_5;\ X_2\cup X_5,\ X_3\cup X_5\rangle$

(VII) $\exp(X)_6=\varnothing$.

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1) The symbol $\rangle A\langle=\{F: F\in\exp(X)$ and $F\cap A\neq\emptyset\}$ and $\langle A_0; A_1,\dots,A_n\rangle=\langle A_0\rangle\cap\rangle A_1\langle\cap\dots\cap\rangle A_n\langle$.

Proof. (I): Let $F\in\exp(X)_0$. Then, $\{F\}$ is open in $\exp(X)$ and there is a basic open set of the form $\langle U_1, U_2, \dots, U_n\rangle$ such that

$$F\in\langle U_1, U_2, \dots, U_n\rangle=\{F\rangle.$$

Let $x_i\in F\cap U_i$, then $\{x_1, x_2, \dots, x_n\}=F$. Let $\{x_{n_1}, \dots, x_{n_k}\}$ be the set of all distinct points of F and $U_{n_1}, \dots, U_{n_k}$ their disjoint and open neighborhoods chosen so that $F\in\langle U_{n_1}, \dots, U_{n_k}\rangle=\{F\}$. Then $U_{n_i}=x_{n_i}$, and the points $x_{n_1}, \dots, x_{n_k}$ are isolated. This proves that $F\subseteq X_0$ or $F\in\langle X_0\rangle$.

If $F\in\langle X_0\rangle$, then F is finite and each of its points is isolated. Let $F=\{x_1, \dots, x_n\}$. Then

$$\{F\}=\langle x_1, \dots, x_n\rangle$$

is open and F is an isolated point of $\exp(X)$.

(II) Let $F\in\exp(X)_1$. Then, by 2.6 $F\in\overline{\exp(X_0)}=\langle\bar{X}_0\rangle$. So $F\cap(X\setminus\bar{X}_0)\neq\varnothing$ or, by 2.6 again, $F\cap X_1\neq\varnothing$, what can be written as $F\in\rangle X_1\langle$.

If $F\in\rangle X_1\langle$, then $\rangle X_1\langle$ is an open neighborhood of F which does not intersect $\langle X_0\rangle$. So $F\notin\overline{\langle X_0\rangle}=\overline{\exp(X)_0}$, what implies $F\in\exp(X)_1$.

(III): To prove (III), we use 2.6, (I) and (II).

$$F\in\exp(X)_2\Rightarrow\begin{cases}F\notin\overline{\exp(X)_1}=\rangle\bar{X}_1\langle\\ \text{and}\\ F\notin\exp(X)_0=\langle X_0\rangle.\end{cases}$$

So we have $F\subseteq X_0\cup X_2$ and then $F\cap X_2\neq\varnothing$. Hence, $F\in\langle X_0\cup X_2, X_2\rangle$.

If $F\in\langle X_0\cup X_2, X_2\rangle$ then $\begin{cases}F\subseteq X_0\cup X_2=X\setminus\bar{X}_1\\ \text{and}\\ F\cap X_2\neq\varnothing\end{cases}$. So we have $F\cap\bar{X}_1=\varnothing$ or $F\notin\rangle\bar{X}_1\langle=\overline{\exp(X)_1}$. Since $F\cap X_2\neq\varnothing$, $F\notin\exp(X)_0$ and so $F\in\exp(X)_2$.

(IV): $F\in\exp(X)_3\Rightarrow\begin{cases}F\notin\overline{\exp(X)_2}=\langle\bar{X}_0, \bar{X}_2\rangle\\ \text{and}\\ F\notin\exp(X)_0,\ F\notin\exp(X)_1\end{cases}$. Note that $\overline{X_0\cup X_2}=$ $=\bar{X}_0=X\setminus X_1$, and if $F\notin\exp(X)_1$ then $F\subseteq\bar{X}_0$ and this condition puts no restriction on F. So $F\cap\bar{X}_2=\varnothing$, what implies $F\subseteq X_0\cup X_3$ since $F\notin\exp(X)_1$. Having $F\notin\exp(X)_0$, then $F\cap X_3\neq\varnothing$. Hence, $F\in\langle X_0\cup X_3, X_3\rangle$.

$F\in\langle X_0\cup X_3, X_3\rangle\Rightarrow\begin{cases}F\subseteq X_0\cup X_3\subseteq X\setminus\bar{X}_2\\ \text{and}\\ F\cap X_3\neq\varnothing.\end{cases}$. Being $F\cap\bar{X}_2=\varnothing$, by (III), $F\notin\overline{\exp(X)_2}$. Since $F\cap X_3\neq\varnothing$, $F\notin\exp(X)_0$ and since $F\subseteq X_0\cup X_3$, $F\notin\exp(X)_1$. Hence, $F\in\exp(X)_3$.

(V):

$$F\in\exp(X)_4\Rightarrow\begin{cases}F\notin\overline{\exp(X)_3}=\langle\bar{X}_0, \bar{X}_3\rangle\\ \text{and}\\ F\notin\exp(X)_t,\ t=0, 1, 2.\end{cases}$$

So $F\cap \bar{X}_3=\varnothing$ and $F\subseteq X_0\cup X_2\cup X_4$ since $F\notin \exp(X)_1$. From $F\notin \exp(X)_0$ and $F\notin \exp(X)_2$, it follows that $F\cap X_4\neq\varnothing$ and we get $F\in\langle X_0\cup X_2\cup X_4,\ X_4\rangle$.

$$F\in\langle X_0\cup X_2\cup X_4,\ X_4\rangle \Rightarrow \begin{cases} F\subseteq X_0\cup X_2\cup X_4\subseteq X\setminus \bar{X}_3 \\ \text{and} \\ F\cap X_4\neq\varnothing \end{cases}.$$ Being $F\cap\bar{X}_3=\varnothing$,

by (IV), $F\notin \overline{\exp(X)_3}$. Since $F\cap X_4\neq\varnothing$, $F\notin\exp(X)_t$, $t=0, 2$ and since $F\cap X_1=\varnothing$, $F\notin\exp(X)_1$. Hence, $F\in\exp(X)_4$.

(VI):
$$F\in\exp(X)_5 \Rightarrow \begin{cases} F\notin\overline{\exp(X)_4}=\langle \bar{X}_0,\ \bar{X}_4\rangle \\ \text{and} \\ F\notin\exp(X)_t,\ t=0, 1, 2, 3. \end{cases}$$

So $F\cap\bar{X}_4=\varnothing$ and since $F\notin\exp(X)_1$, we have $F\subseteq X_0\cup X_2\cup X_3\cup X_5$. Then, $F\notin\exp(X)_0$ implies $F\cap(X_2\cup X_3\cup X_5)\neq\varnothing$, $F\notin\exp(X)_2$ implies $F\cap(X_3\cup X_5)\neq\varnothing$ and $F\notin\exp(X)_3$ implies $F\cap(X_2\cup X_5)\neq\varnothing$ and since $X_2\cup X_3\cup X_5=(X_2\cup X_5)\cup\cup(X_3\cup X_5)$, the latter two conditions imply the former. Hence, we have

$$F\in\langle X_0\cup X_2\cup X_3\cup X_5,\ X_2\cup X_5,\ X_3\cup X_5\rangle.$$

$$F\in\langle X_0\cup X_2\cup X_3\cup X_5,\ X_2\cup X_5,\ X_3\cup X_5\rangle$$
$$\Rightarrow \begin{cases} F\subseteq X_0\cup X_2\cup X_3\cup X_5\subseteq X\setminus\bar{X}_4 \\ \text{and} \\ F\cap(X_2\cup X_5)\neq\varnothing,\ F\cap(X_3\cup X_5)\neq\varnothing. \end{cases}$$

Being $F\cap\bar{X}_4=\varnothing$, by (V), $F\notin\overline{\exp(X)_4}$ and since $F\cap X_1=\varnothing$, $F\notin\exp(X)_1$. $F\cap(X_2\cup X_5)\neq\varnothing$ implies that $F\notin\exp(X)_0$ and $F\notin\exp(X)_3$. $F\cap(X_3\cup X_5)\neq\varnothing$ implies $F\notin\exp(X)_2$, Hence, $F\in\exp(X)_5$.

(VII): Suppose there exists an $F\in\exp(X)_6$. Then $F\notin\overline{\exp(X)_5}$ and $F\notin\exp(X)_t$, $t=0, 1, 2, 3, 4$. By (VI), $F\notin\langle\bar{X}_0,\ \bar{X}_2,\ \bar{X}_3\rangle$ what means that either $F\cap\bar{X}_2=\varnothing$ or $F\cap\bar{X}_3=\varnothing$. In the first case, $F\subseteq X_0\cup X_3$, what is impossible since $F\notin\exp(X)_3$. In the second case $F\subseteq X_0\cup X_2\cup X_4$ since $F\notin\exp(X)_1$. But $F\notin\exp(X)_4$ so that $F\cap X_4=\varnothing$. Then $F\subseteq X_0\cup X_2$. The last inclusion contradicts $F\notin\exp(X)_0$ and $F\notin\exp(X)_1$. Hence there exists no $F\in\exp(X)$ being in $\exp(X)_6$. Therefore, $\exp(X)_6=\varnothing$.

3.2. *For each* X, $\exp(X)_n=\varnothing$, *for* $n=8, 9, \ldots$ *If* $X_3\neq\varnothing$ *and* $X_4\neq\varnothing$, *then* $\exp(X)_7\neq\varnothing$, *otherwise* $\exp(X)_7=\varnothing$.

Proof. If $X_3=\varnothing$, then $X_n=\varnothing$ for $n=5, 6, \ldots$ and $X=X_0\cup X_1\cup X_2\cup X_4$. Let $F\in\exp(X)$, if $F\cap X_1\neq\varnothing$ then $F\in\exp(X)_1$, if $F\subseteq X_0\cup X_2\cup X_4$ then F belongs to one of the sets $\exp(X)_0$, $\exp(X)_2$, $\exp(X)_4$. So $\exp(X)_5=\varnothing$ and then $\exp(X)_7=\varnothing$.

If $X_4=\varnothing$, then $X_n=\varnothing$ for $n=6, 7, \ldots$ and $X=X_0\cup X_1\cup X_2\cup X_3\cup X_5$. We see again that every $F\in\exp(X)$ belongs to one of the sets $\exp(X)_t$, for $t=0, 1, 2, 3, 5$. We have again $\exp(X)_7=\varnothing$.

If $X_3\neq\varnothing$ and $X_4\neq\varnothing$, then $\langle X_3,\ X_4\rangle\neq\varnothing$ and $\langle X_3,\ X_4\rangle\subseteq\exp(X)_7$, since no $F\in\exp(X)_t$, $t=0, 1, 2, 3, 4, 5$, belongs to $\langle X_3,\ X_4\rangle$.

3.3. *For* $t=1, 2, 3, 4, 5, 7$, $\exp(X)_t$ *is either empty or homeomorphic to* C.

Proof. Let $\overline{\exp(X)_1} \neq \varnothing$. Then, by 3.1, $\overline{\exp(X)_1} = \rangle \bar{X}_1 \langle$ and $\bar{X}_1 \neq \varnothing$. By 2.3, $\bar{X}_1 \approx C$. Let $F \in \rangle \bar{X}_1 \langle$ and let $\langle U_1, \ldots, U_n \rangle$ be an open neighborhood of F in $\exp(X)$. Since $\bar{X}_1 \cap F \neq \varnothing$, let $x_0 \in \bar{X}_1 \cap F$ and let $x_i \in U_i \cap F$, $i = 1, \ldots, n$. Then the set $\{x_0, x_1, \ldots, x_n\} \in \langle U_1, \ldots, U_n \rangle$ and $x_0 \in U_{i_0}$ for some i_0. Since $\bar{X}_1 \approx C$, there will exist infinitly many points of $\bar{X}_1$ in U_{i_0}. For each point $x \in U_{i_0}$, $\{x, x_1, \ldots, x_n\} \in \langle U_1, \ldots, U_n \rangle$. Hence F is not an isolated point in $\overline{\exp(X)_1}$ and then $\overline{\exp(X)_1} \approx C$.

By 3.1, $\overline{\exp(X)_t} = \langle \bar{X}_0, \bar{X}_t \rangle$ for $t = 2, 3, 4$. Let $\overline{\exp(X)_t} \neq \varnothing$ and $F \in \overline{\exp(X)_t}$. Then, $\bar{X}_t \neq \varnothing$ and $\bar{X}_t \subseteq \bar{X}_0$. Let $x_0 \in F \cap \bar{X}_t$, then x_0 is not an isolated point of X and there exists a sequence (x_n) of points from X_0 such that $x_n \to x_0$. Then $F_n = F \cup \{x_n\} \to F$ in $\exp(X)$ and $F_n \in \overline{\exp(X)_t}$ for every n. If infinitly many of the sets F_n are different from F, then we have proved that F is not an isolated point in $\exp(X)_t$. In case that $F_n = F$ for almost all n, let us form another sequence (H_n), where $H_n = F \setminus \{x_n\}$. The sets H_n are closed since the points x_n are isolated and since $x_0 \in \bar{X}_t$, $x_n \neq x_0$ for every n. It is easily seen that $H_n \in \overline{\exp(X)_t}$ and that $H_n \to F$ in $\exp(X)$. So in any case, $\overline{\exp(X)_t}$ has no isolated point and $\overline{\exp(X)_t} \approx C$.

$\overline{\exp(X)_5} = \langle \bar{X}_0, \bar{X}_2, \bar{X}_3 \rangle$ and if $\overline{\exp(X)_5} \neq \varnothing$, then $\bar{X}_2 \neq \varnothing$ and $\bar{X}_3 \neq \varnothing$. Let $F \in \overline{\exp(X)_5}$ and let $x_0 \in F \cap \bar{X}_2$. Since $\bar{X}_2 \subseteq \bar{X}_0$, there exists a sequence (x_n) of isolated points of X such that $x_n \to x_0$. Now we conclude again that at least one of these two sequences $(F \setminus \{x_n\})$, $(F \cup \{x_n\})$ converging to F, has infinitly many members being different from F and all their members belong to $\overline{\exp(X)_5}$. Hence, $\overline{\exp(X)_5} \approx C$.

Suppose $F \in \exp(X)_7$. Then F has a point x_0 belonging to $\bar{X}_0 \setminus X_0$. Let (x_n) be the sequence of isolated points of X converging to x_0. Then again, at least one of the sequences $(F \cup \{x_n\})$, $(F \setminus \{x_n\})$ converging to F, has infinitly many members different from F. Using 3.1, it is easy to see that if $F \notin \exp(X)_t$, $t = 0, 1, 2, 3, 4, 5$ then neither $F \setminus \{x_n\}$ nor $F \cup \{x_n\}$ belong to $\exp(X)_t$, $t = 0, 1, 2, 3, 4, 5$, for any n. Therefore, $\exp(X)_7$ has no isolated point and so $\exp(X)_7 \approx C$.

4. $\exp(X)$ for $X \in \mathcal{Z}$. We know from 3.3 that $\overline{\exp(X)_t}$ is empty from $t = 8$ on, and if $\overline{\exp(X)_t}$ is not empty, then for $t = 1, 2, 3, 4, 5, 7$ $\overline{\exp(X)_t} \approx C$. The definitions that follow are subjected to this fact.

Call a space X *full*, whenever $X_n \neq \varnothing$ implies $\bar{X}_n \approx C$, for $n = 1, 2, \ldots$ Accordingly, every space X having isolated points only (and then a finite number of them) is full, C is full as well as topological union of such two spaces.

According to 2.7, we can associate with every space X the sequence of numbers being the accumulation orders of its points. In that way we obtain sequences of the form

$$(0),\ (\varnothing, 1),\ (0, 1),\ (0, \varnothing, 2),\ (0, 1, 2),\ \ldots$$

$$(0, 1, 2, \ldots, n-2, \varnothing, n),\ (0, 1, 2, \ldots, n-1, n),\ \ldots$$

$$(0, 1, 2, \ldots, n, \ldots,).$$

The empty set stands to denote that $X_{n-1} = \varnothing$, $X_n \neq \varnothing$ and $X_{n+k} = \varnothing$, $k = 1, 2, \ldots$ The sequence so associated with the space X will be denoted by $s(X)$ and called the *accumulation spectrum* of the space X. Call $s(X)$ *finite* if it is a finite sequence.

Let X and Y be two full spaces such that $s(X)$ and $s(Y)$ are finite. Then, say that X and Y are *equivalent* if $s(X)=s(Y)$ and $\operatorname{card}(X_0)=\operatorname{card}(Y_0)$.

4.1. *Let X and Y be equivalent. If X is decomposed into the union of two disjoint, closed and open subsets X' and X'', then Y can be decomposed into the union of two disjoint, closed and open subsets Y' and Y'' in such a manner that X' is equivalent to Y' and X'' to Y''.*

Proof. First of all, let us note that if X and Y are homeomorphic, then they are equivalent and if $h: X \to Y$ is a homeomorphism, then $Y'=h(X')$ and $Y''=h(X'')$. So in case of homeomorphic spaces X and Y, 4.1 holds.

On the other hand, by 2.8, $Xt=(X')_t+(X'')_t$ and if $t \geqslant 1$, then $(X')_t$ and $(X'')_t$ are clopen subsets of X_t. Being X full, $X_t \approx C$ so that the spaces $(X')_t$ and $(X'')_t$ are either empty or homeomorphic to C. So we conclude that a clopen subset of a full space is full itself.

If $s(X)=(0)$, then $s(Y)=(0)$ and $X=X_0 \approx Y=Y_0$, so that 4.1 holds.

If $s(X)=(\varnothing, 1)$, then $X \approx Y \approx C$ and 4.1 holds again.

If $s(X)=(0, 1)$, then $X=X_0+X_1$ and $Y=Y_0+Y_1$, $X_0 \approx Y_0$, $X_1 \approx Y_1 \approx C$. Hence, $X \approx Y$ and 4.1 holds.

Suppose now $s(X)=(\ldots, \varnothing, n)$, with $n \geqslant 2$. Evidently X_0 is infinite, in fact $\operatorname{card}(X_0)=\aleph_0$. By 2.8, $X_n=(X')_n+(X'')_n$ and at least for one of the spaces X', X'' the set of its n-points will be non-empty. Let it be X'. Then, $s(X')=(\ldots, \varnothing, n)$ and $s(X'')$ is either $(\ldots, \varnothing, m)$ or $(\ldots, m-1, m)$, $m \leqslant n$ and we take $(\varnothing, 0)$ to be (0). So we have the following two cases

(a) $s(X')=(\ldots, \varnothing, n)$ and $s(X'')=(\ldots, \varnothing, m)$

(b) $s(X')=(\ldots, \varnothing, n)$ and $s(X'')=(\ldots, m-1, m)$.

Consider the case (a). If $m=0$, then X'' is a finite subset of X_0 and let Y'' be any subset of Y_0 such that $\operatorname{card}(X_0)=\operatorname{card}(Y_0)$. Put $Y'=Y-Y''$. Then $X'' \approx Y''$ and Y' contains all n-points of Y. So $s(Y')=(\ldots, \varnothing, n)$ and X' is equivalent to Y'.

If $m>0$, let y be a point of Y_m. Take Y'' to be a clopen neighborhood of y such that $Y_m \setminus Y'' \neq \varnothing$ and $Y'' \cap \overline{Y}_{m-1}=\varnothing$. Then, $s(Y'')=(\ldots, \varnothing, m)$ and since for $m=n$, $Y_n \setminus Y'' \neq \varnothing$ and for $m<n$, $Y_n \subseteq \overline{Y}_{m-1}$, it follows that $(Y \setminus Y'')_n \neq \varnothing$. Putting $Y'=Y \setminus Y''$ we have $s(Y')=(\ldots, \varnothing, n)$. Hence, X' is equivalent to Y' and X'' to Y''.

Consider the case (b). If $m=1$, then $s(X'')=(0, 1)$ and $X''=X_0''+X_1''$. Let U be a subset of Y_0 such that $\operatorname{card}(U)=\operatorname{card}(X_0'')$ and V a clopen subset of Y_1, then $Y''=U+V$ is equivalent to X''. Put $Y'=Y \setminus Y''$, then $s(Y')=$ $=(\ldots, \varnothing, n)$ so that Y' is equivalent to X'. If $m>1$, let U be a clopen neighborhood of a point in Y_{m-1} such that $U \subseteq Y_0 \cup \cdots \cup Y_{m-1}$ and V a clopen neighborhood of a point in Y_m such that $V \cap \overline{Y}_{m-1}=\varnothing$. Let $Y''=U+V$, then $Y'' \cap Y_n=\varnothing$ (m must be less than n, since $Y_{n-1}=\varnothing$ and $Y_{m-1} \neq \varnothing$). Now we have $s(Y'')=(\ldots, m-1, m)$ and putting $Y'=Y \setminus Y'$, $s(Y')=(\ldots, \varnothing, n)$. Therefore, X' is equivalent to Y' and X'' to Y''.

Now we have left to consider the case when $s(X)=(\ldots, n-1, n)$, for $n \geqslant 2$. From the relation $X_n=(X')_n+(X'')_n$, we can suppose that $s(X')=(\ldots, \varnothing, n)$ or $s(X')=(\ldots, n-1, n)$ and there will be four possibilities

(c) $s(X')=(\ldots, \varnothing, n)$ $\quad s(X'')=(\ldots, n-2, n-1)$

(d) $s(X')=(\ldots, \varnothing, n)$ $\quad s(X'')=(\ldots, \varnothing, n-1)$

(e) $s(X')=(\ldots, n-1, n)$ $\quad s(X'')=(\ldots, \varnothing, m)$

(f) $s(X')=(\ldots, n-1, n)$ $\quad s(X'')=(\ldots, m-1, m)$.

Consider the case (c). Let U be a clopen neighborhood of Y_{n-1} ($\bar{Y}_{n-1}=Y_{n-1}$) which does not intersect Y_n. Let V be a clopen neighborhood of a point in Y_{n-2} such that $V \subseteq Y_0 \cup \cdots \cup Y_{n-2}$. Then, $Y''=U+V$ is such that $s(Y'')=(\ldots, n-2, n-1)$ and putting $Y'=Y \setminus Y''$, $s(Y')=(\ldots, \varnothing, n)$.

Consider the case (d). Let U be a clopen neighborhood of Y_{n-1} which does not intersect $\bar{Y}_{n-2}=Y_{n-2} \cup Y_n$. Put $Y'=U$, $Y''=Y \setminus Y'$, then $s(Y')=(\ldots, \varnothing, n)$, $s(Y'')=(\ldots, \varnothing, n-1)$.

Consider the case (e). If $m=0$, let Y'' be a set of points of Y_0 such that $\operatorname{card}(Y'')=\operatorname{card}(X'')$. Put $Y'=Y \setminus Y''$ and $s(Y'')=(\ldots, n-1, n)$. If $m>0$, let y be a point of Y_m and Y'' a clopen neighborhood of y such that $Y_m \setminus Y'' \neq \varnothing$ and $Y'' \cap \bar{Y}_{m-1}=\varnothing$. Then, $s(Y'')=(\ldots, \varnothing, m)$ and $s(Y')=(\ldots, n-1, n)$.

Consider the case (f). If $m=1$, then $X''=X''_0+X''_1$. Let U be a subset of Y_0 such that $\operatorname{card} U=\operatorname{card} X''_0$ and V a clopen subset of Y_1. Put $Y''=U+V$, $Y'=Y \setminus Y''$. Then $s(Y'')=(0, 1)$ and $s(Y')=(\ldots, n-1, n)$. If $m>1$, Let U be a clopen neighborhood of a point in Y_m such that $Y_m \setminus U \neq \varnothing$ and $U \cap \bar{Y}_{m-1}=\varnothing$. Let V be a clopen neighborhood of a point in Y_{m-1} such that $Y_{m-1} \setminus V \neq \varnothing$ and $V \subseteq Y_0 \cup \cdots \cup Y_{m-1}$. Put $Y'=U+V$ and $Y''=Y \setminus Y'$. Then, it is easy to see that $s(Y')=(\ldots, n-1, n)$ and $s(Y'')=(\ldots, m-1, m)$.

We have seen that in all possible cases Y can be decomposed as it is stated in 4.1.

Our next statement strengthens 4.1 and shows that two equivalent spaces are homeomorphic.

4.2. *If X and Y are equivalent, then they are homeomorphic.*

P r o o f. Both X and Y can be considered as subspaces of the Cantor discontinuum C and that is why we suppose that they are subsets of the interval $[0, 1]$. Decompose X into two clopen subsets X^1 and X^2 so that $\operatorname{diam}(X^i)<2/3$ ($\operatorname{diam}(X) \leqslant 1$). By 4.1, there is a decomposition of Y into two clopen subsets Y^1 and Y^2 so that Y^i is equivalent to X^i. Using again 4.1, we can decompose Y^i ($i=1, 2$) into two clopen subsets Y^{i1} and Y^{i2} in such a way that $\operatorname{diam}(Y^{ik})<2/3$ ($\operatorname{diam}(Y)=1$). Let X^{i1} and X^{i2} be the corresponding pair representing the decomposition of X^i so that X^{ik} is equivalent to Y^{ik}. Evidently, $\operatorname{diam}(X^{ik})<2/3$. If some of the sets Y^i is a one-point set, let $Y^{i1}=Y^i$, $Y^{i2}=\varnothing$. Then, X^i is also a one-point set and we put $X^{i1}=X^i$, $X^{i2}=\varnothing$. In this way we get two coverings for each of the spaces X and Y. Let

$$\mathcal{U}^1=\{X^{11}, X^{12}, X^{21}, X^{22}\}, \quad \mathcal{V}^1=\{Y^{11}, Y^{12}, Y^{21}, Y^{22}\}.$$

Suppose the coverings

$$\mathcal{U}^n=\{X^{ij\cdots r}\}, \quad \mathcal{V}^n=\{Y^{ij\cdots r}\}$$

have been constructed, where $ij \dots r$ is a sequence of $2n$ numbers each being 1 or 2. Suppose that

$$\operatorname{diam}(X^{ij \dots r}) < (2/3)^n, \quad \operatorname{diam}(Y^{ij \dots r}) < (2/3)^n,$$

and $X^{ij \dots r}$ and $Y^{ij \dots r}$ are disjoint, equivalent clopen subsets of X and Y respectively. If $X^{ij \dots r}$ is one point, put $X^{ij \dots r1} = X^{ij \dots r}$, $Y^{ij \dots r1} = Y^{ij \dots r}$ and if not, let $X^{ij \dots r1}$ and $X^{ij \dots r2}$ be two clopen subsets of $X^{ij \dots r}$ such that their diameters are less than $2/3 \operatorname{diam}(X^{ij \dots r})$. Let $Y^{ij \dots r1}$ and $Y^{ij \dots r2}$ be the clopen subsets into which $Y^{ij \dots r}$ is decomposed and let they be equivalent to $X^{ij \dots r1}$ and $X^{ij \dots r2}$ respectively. If $Y^{ij \dots rs}$ $(s = 1, 2)$ is one point let $Y^{ij \dots rs1} = Y^{ij \dots rs}$ and $X^{ij \dots rs1} = X^{ij \dots rs1}$ and if not, let $Y^{ij \dots rs}$ is decomposed into two clopen subsets $Y^{ij \dots rst}$ $(t = 1, 2)$ such that $\operatorname{diam}(Y^{ij \dots rst}) < 2/3 \operatorname{diam}(Y^{ij \dots rs})$. Let $X^{ij \dots rs}$ is decomposed into two clopen subsets $X^{ij \dots rst}$ $(t = 1, 2)$ such that $X^{ij \dots rst}$ is equivalent to $Y^{ij \dots rst}$. Let

$$\mathcal{U}^{n+1} = \{X^{ij \dots rst}\}, \quad \mathcal{V}^{n+1} = \{Y^{ij \dots rst}\},$$

then we get two coverings $\mathcal{U}^{n+1}$, $\mathcal{V}^{n+1}$ of the spaces X, Y having for their members disjoint clopen subsets and being $X^{ij \dots rst}$ equivalent to $Y^{ij \dots rst}$ and

$$\operatorname{diam}(X^{ij \dots rst}) < (2/3)^{n+1}, \quad \operatorname{diam}(Y^{ij \dots rst}) < (2/3)^{n+1}.$$

Let $f_n : \mathcal{U}^{n+1} \to \mathcal{V}^n$ and $g_n : \mathcal{V}^{n+1} \to \mathcal{U}^n$ be given by

$$f_n(X^{ij \dots rst}) = X^{ij \dots r}, \quad g_n(Y^{ij \dots rst}) = Y^{ij \dots r},$$

and thinking of the members of $\mathcal{U}^n$ and $\mathcal{V}^n$, $(n = 1, 2, \dots)$ as points of the discrete spaces $\mathcal{U}^n$ and $\mathcal{V}^n$, we obtain the following two inverse systems

$$\{\mathcal{U}^n, f_n\}, \quad \{\mathcal{V}^n, g_n\}.$$

Being these two systems the same (up to the different letters used to denote the spaces and mappings), their inverse limits are homeomorphic. As it is very well known each of these inverse limits is homeomorphic to its corresponding space (see [1], p. 98). Therefore, X and Y are homeomorphic spaces.

4.3. *All spaces $C_0, C_1, \dots, C_n, \dots$ are full, as well as the spaces $C_{n-1} + C_n$, $(n = 1, 2, \dots)$.*

Proof. In proving 2.9, we have seen that $(C_n)_n \approx C$ and $(C_n)_{n-1} = \varnothing$ for $n \geqslant 1$. It is easy to see that C_0, C_1, C_2 are full. To show that C_n is full for $n \geqslant 3$, we have to prove that $\overline{(C_n)_t} \approx C$. Suppose that the last relation holds for all $n \leqslant k$. Using this induction hypothesis we will show that $\overline{(C_{k+1})_t} \approx C$ for $t = 1, \dots, k-1$. Since

$$C_{k+1} = (C_k \times \{0\}) \cup \left(\bigcup_{m=1}^{\infty} Q_{k-1}^m \right),$$

where $Q_{k-1}^m = i_{k-1}(C_{k-1}^m) \times \left\{ \frac{1}{m} \right\}$. If $x \in (C_{k+1})_t$ then $x \notin (C_{k+1})_{k+1}$ and $x \in \in (C_{k-1}^m) \times \left\{ \frac{1}{m} \right\}$ for some m, or $x \in (C_k \times \{0\}) \setminus (C_k)_k \times \{0\}$. In both cases x is a t-point having a clopen neighborhood contained in these sets which are parts of homeomorphic images of C_{k-1} and C_k. By the induction hypothesis, x is not an isolated point in the set of t-points of these subsets and so x is not an isolated point of $(C_{k+1})_t$. Hence, $\overline{(C_{k+1})_t} \approx C$.

Applying 2.8, now it is easy to see that $C_{n-1} + C_n$ are full.

4.4. *Let X be a full space. If $s(X) = (\dots, \varnothing, n)$ then $X \approx C_n$. If $s(X) = (\dots, n-1, n)$, then $X \approx C_{n-1} + C_n$.*

P r o o f. The accumulation spectrum of C_n is $(\dots, \varnothing, n)$ and of $C_{n-1} + C_n$, $(\dots, n-1, n)$. Now 4.4 follows from 4.2 and 4.3.

Let us note that $C_i + C_j \approx C_j$, if $i \leqslant j-2$. This follows from $(C_i)_t + (C_j)_t = (C_j)_t$ for $t = j-1, j$, and so $C_i + C_j$ is full and $s(C_i + C_j) = (\dots, \varnothing, j)$.

We know from 3.3 that $\exp(X)$ is a full space. Using 3.1 and 3.2 and according to 4.4, we will have

4.5. (a) $s(X) = (0)$, $\exp(X) = \langle X \rangle$

(b) $s(X) = (1)$, $\exp(X) \approx C_1$

(c) $s(X) = (0, 1)$, $\exp(X) \approx \langle X_0 \rangle + C_1$

(d) $s(X) = (0, \varnothing, 2)$, $\exp(X) \approx C_2$

(e) $s(X) = (0, 1, 2)$, $\exp(X) \approx C_1 + C_2$

(f) $s(X) = (0, 1, \varnothing, 3)$, $\exp(X) \approx C_3$

(g) $s(X) = (0, 1, 2, 3)$, $\exp(X) \approx C_5$

(h) $s(X) = (0, 1, 2, \varnothing, 4)$, $\exp(X) \approx C_4$

(i) $s(X) = (0, 1, 2, 3, \varnothing, 5)$, $\exp(X) \approx C_5$

(j) for every other X, $\exp(X) \approx C_7$.

Excluding two cases (a) and (c) in 4.5, there exist seven different topological types of hyperspaces of the spaces from $\mathcal{Z}$. In case a space is exponentially complete, then it must be one of the spaces indicated in 4.5. If X is such that $s(X) = (0)$ or $s(X) = (0, 1)$ then X is exponentially complete only in case when X_0 is a one-point set. Now we have.

4.6. *There exist exactly nine exponentially complete spaces in $\mathcal{Z}$ and they are*

$$C_0,\ C_1,\ C_2,\ C_3,\ C_4,\ C_5,\ C_7,\ C_0 + C_1,\ C_1 + C_2.$$

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