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ADDITIONS TO KAMKE'S TREATISE II: NONLINEAR FIRST AND SECOND ORDER DIFFERENTIAL EQUATIONS

Jovan D. Kečkić

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A great majority of differential equations in Kamke's collection [1] (see also [2]) are of the form

$$\sum_{i=1}^{m} a_i x^{\alpha_i} \prod_{k=0}^{n_i} (y^{(k)}(x))^{\beta_k} = 0,$$

i.e. a sum of a number of terms of the form $ax^{\alpha} y^{\beta} y'^{\gamma} \cdots (y^{(k')})^{\nu}$ is equated to zero.

Probably for that reason D. S. Mitrinović [3] (see also [2], pp. 558—559) examined the differential equation

$$x^{\alpha} y^{\beta} (y')^{\gamma} + Ay^{\lambda} (y')^{\mu} + Bx^{\nu} = 0 \tag{1}$$

($x>0$, $y>0$, $y'>0$; α, β, γ, λ, μ, ν, A, B are real constants such that $AB \neq 0$, $|\alpha|+|\nu|>0$, $|\beta|+|\lambda|>0$, $|\gamma|+|\mu|>0$) and came to the conclusion that if

$$\beta+\gamma\neq 0, \quad (\beta+\gamma)(\mu+\nu)+(\lambda+\mu)(\alpha-\gamma-\nu)=0, \tag{2}$$

then equation (1) can be reduced to a first order equation which does not contain the independent variable, and which can, thus, in certain cases be integrated by quadratures. D. S. Mitrinović also noted that Kamke's equations 1.385, 1.386, 1.397, 1.404, 1.413, 1.414, 1.415, 1.455, 1.457. 1.470, 1.476, 1.487, 1.525, 1.527, 1.541, 1.542, 1.550, 1.554. are special cases of (1) and that they satisfy (2).

In this paper we shall go a step further.

Theorem. *Let $F(x_1, x_2, x_3)$ be a homogeneous function, i.e.*

$$F(tx_1, tx_2, tx_3) = t^{\alpha} F(x_1, x_2, x_3) \qquad (\alpha \neq 0). \tag{3}$$

Then the differential equation

$$F(x^k, y, xy') = 0 \qquad (k \neq 0) \tag{4}$$

can be transformed to the form

$$f(u, u') = 0. \tag{5}$$

Integration of equation (4) is reduced in that case to the question of solving $f(u, u')=0$ with respect to u' and to one quadrature.

Proof. Let

$$x=e^t, \quad y=e^{kt}u(t). \tag{6}$$

We then have $y'=e^{(k-1)t}(ku+u')$ and equation (4) becomes

$$F(e^{kt},\ e^{kt}u,\ e^{kt}(ku+u'))=0,$$

or, in virtue of (3),

$$e^{kt}F(1,\ u,\ ku+u')=0$$

Putting $F(1,\ u,\ ku+u')\equiv f(u, u')$, we arrive at equation (5).

The surprising fact is that 68 first order equations from Kamke's collection can be integrated in this way. We shall give here more general equations which contain all those 68 Kamke's equations (as well as many more other equations). In each case we specify the constant k from (6) and state the numbers of Kamke's equations which they contain.

1. $ax^{\alpha-1}y^2y'+bx^{\alpha}y'+cy^{2\alpha-1}+dx^{\alpha-1}y=0 \qquad k=\dfrac{1}{2}.$

Kamke's equations 1.277, 1.284, 1.293, 1.301.

2. $y'+ax^{\alpha}y^{\beta}+bx^{\gamma}y^{\delta}=0 \quad \left(\delta=\dfrac{\alpha+\beta\gamma+\beta-\gamma}{\alpha+1},\ \alpha\neq-1,\ \beta\neq1\right) \qquad k=\dfrac{\alpha+1}{1-\beta}.$

Kamke's equations 1.41, 1.52, 1.58, 1.143, 1.189.

3. $axyy'+bx^2y'+cy^2+dxy+ex^2=0 \qquad k=1.$

Kamke's equation 1.246.

4. $(ax^{\alpha}y^{\beta}+bx^{\gamma}y^{\delta}+c)xy'+(dx^{\alpha}y^{\beta}+ex^{\gamma}y^{\delta}+f)y=0 \ (\alpha(1-\delta)=\gamma(1-\beta),\ \beta\neq\delta)$

$k=\dfrac{\gamma-\alpha}{\beta-\delta}.$

Kamke's equations 1.302, 1.303, 1.304, 1.329, 1.333.

5. $axy'^2+byy'+cx^{\alpha}y^{\beta}+dx^{\gamma}=0 \quad \left(\gamma=\dfrac{\beta+2\alpha}{2-\beta},\quad \beta\neq2\right) \qquad k=\dfrac{\alpha+1}{2-\beta}.$

Kamke's equations 1.407, 1.408, 1.413, 1.414, 1.415, 1.418, 1.419, 1.421, 1.422, 1.423, 1.424, 1.434, 1.438, 1.439, 1.440, 1.455.

6. $ay^{\alpha}y'^2+bx^{\beta}y'+cx^{\beta-1}y=0 \qquad (\alpha\neq-1) \qquad k=\dfrac{\beta+1}{\alpha+1}.$

Kamke's equations 1.385, 1.386, 1.397, 1.398, 1.400, 1.401, 1.404, 1.411, 1.457, 1.464, 1.465, 1.466, 1.467, 1.468, 1.469, 1.470, 1.475, 1.476, 1.487.

7. $ay^{\alpha}y^2+bx^2y^2+cxyy'+dy^2+ex^{\beta}y^{\gamma}=0 \quad \left(\gamma=\dfrac{4-\alpha\beta}{2},\quad \alpha\neq0\right) \qquad k=\dfrac{2}{\alpha}.$

Kamke's equations 1.402, 1.453, 1.454, 1.480.

8. $ayy'^2 + byy' + cxy' + dx = 0 \qquad k = 1$

Kamke's equation 1.471.

9. $ay'^3 + bxy^{\alpha} y' + cy^{\alpha+1} = 0 \qquad (\alpha \neq 2) \qquad k = \dfrac{3}{2-\alpha}.$

Kamke's equations 1.525, 1.527, 1.541, 1.542.

10. $ax^3 y'^3 + bx^2 yy'^2 + cxy^2 y' + dx^{\alpha+1} y' + ey^3 + fx^{\alpha} y + hx^{6/\alpha} = 0 \qquad (\alpha \neq 0)$

$k = \dfrac{2}{\alpha}.$

Kamke's equations 1.534, 1.535, 1.537.

11. $ayy'^3 + byy'^2 + cxy' + dx = 0 \qquad k = 1.$

Kamke's equation 1.540.

12. $y'^r = ay^s + bx^{rs/(r-s)} \qquad (r \neq s) \qquad k = \dfrac{r}{r-s}.$

Kamke's equation 1.550.

13. $ax^{n-1} y'^n + axy' + by = 0 \qquad (n \neq 1) \qquad k = \dfrac{1}{n-1}.$

Kamke's equation 1.554.

It should be noticed that the above equations present considerable generalisations of Kamke's equations, since their coefficients are arbitrary real numbers, while in Kamke's collection the coefficients are (almost in all cases) special real numbers.

Remark. Since the general solution of the functional equation (3) is given by

$$F(x_1, x_2, x_3) = x_1^{\alpha} G\left(\frac{x_2}{x_1}, \frac{x_3}{x_1}\right),$$

where G is an arbitrary function, we see that equations of the form $G(x^r y, x^{r+1} y') = 0$ can be integrated by the above method.

For example, returning to Mitrinović's equation (1)

$$x^{\alpha-\nu} y^{\beta} y'^{\gamma} + Ax^{-\nu} y^{\lambda} y'^{\mu} + B = 0,$$

we see that it must be of the form

$$(x^r y)^{\beta} (x^{r+1} y')^{\gamma} + A (x^r y)^{\lambda} (x^{r+1} y')^{\mu} + B = 0$$

which implies

$$r\beta + (r+1)\gamma = \alpha - \nu, \quad r\lambda + (r+1)\mu = -\nu$$

or, eliminating r, (2).

The above method can be extended to higher order equations. In that case substitution (6) reduces the order of the equation. The result is as follows:

Let $F(x_1, x_2, \ldots, x_m)$ *be a homogeneous function, i.e.*

(7) $$F(tx_1, tx_2, \ldots, tx_m) = t^{\alpha} F(x_1, x_2, \ldots x_m).$$

Then the differential equation

$$F(x^k, y, xy', \dots, x^n y^{(n)}) = 0 \qquad (n = m - 2)$$

can be reduced to the form

$$f(u, u', \dots, u^{(n)}) = 0,$$

and afterwards to the form

$$g(\xi, \eta, \eta', \dots, \eta^{(n-1)}) = 0 \qquad (\eta = \eta(\xi)).$$

There are 40 second order nonlinear equations in Kamke's collection whose order can be decreased by this method. They are equations 6.11, 6.58, 6.73, 6.74, 6.79, 6.81, 6.87, 6.88, 6.90, 6.93, 6.96, 6.97, 6.98, 6.99, 6.100, 6.102, 6.105, 6.106, 6.133, 6.134, 6.169, 6.172, 6.173, 6.175, 6.176, 6.179, 6.181, 6.182, 6.183, 6.184, 6.189, 6.193a, 6.194, 6.195, 6.205, 6.208, 6.227, 6.229, 6.231 and 6.245. Some of these equations can be considerably generalised. For example, equation

$$axy'' + by' + cx^\nu y^n y'^m = 0 \qquad (\nu - m + 1 \neq 0)$$

contains 12 Kamke's equations, and is much more general in form than they are. In all the above 40 equations coefficients could be taken to be arbitrary real numbers, and not fixed special numbers, as is the case in Kamke.

Remark. As in the case of first order equations, noting that the general solution of equation (7) is given by

$$F(x_1, x_2, \dots, x_m) = x_1^\alpha G\left(\frac{x_2}{x_1}, \dots, \frac{x_m}{x_1}\right)$$

we conclude that the general form of differential equations whose order can be decreased by the described method is

$$G(x^r y, x^{r+1} y', \dots, x^{r+n} y^{(n)}) = 0. \tag{8}$$

The only linear homogeneous differential equation of the form (8) is Euler's equation

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \dots + a_1 xy' + a_0 y = 0.$$

Equation (8) can therefore be regarded as the nonlinear analogy of Euler's equation.

REFERENCES

[1] E. Kamke, *Differentialgleichungen — Lösungsmethoden und Lösungen*, Leipzig 1959.

[2] Э. Камке, *Справочник по обыкновенным дифференциальным уравнениям*, Москва 1971.

[3] D. S. Mitrinović, *Compléments au Traité de Kamke* III, Bol. Un. Mat. Ital. (3) **11** (1956), 168—171.