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SOME INEQUALITIES FOR SPECIAL FUNCTIONS

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1. Inequality

$$\frac{(x-1)^x (y-1)^y}{\left(\frac{x+y}{2}-1\right)^{x+y}} \leqslant \frac{\Gamma(x)\Gamma(y)}{\left(\Gamma\left(\frac{x+y}{2}\right)\right)^2} \qquad (x>1,\ y>1) \tag{1}$$

was posed as problem in [1].

However, in [2], among other things, the following inequalities

$$\frac{x^x y^y}{\left(\frac{x+y}{2}\right)^{x+y}} \leqslant \frac{\Gamma(x)\Gamma(y)}{\left(\Gamma\left(\frac{x+y}{2}\right)\right)^2} \leqslant \frac{(x-1)^{x-1}(y-1)^{y-1}}{\left(\frac{x+y}{2}-1\right)^{x+y-2}} \qquad (x>1,\ y>1) \tag{2}$$

were proved.

We shall show that the left inequality of (2) is sharper than (1). Consider the function f, defined by

$$f(x)=\left(1-\frac{1}{x}\right)^x \qquad (x>1).$$

We have

$$(\log f(x))''=-\frac{1}{x(x-1)^2}<0 \qquad (x>1).$$

The function f is therefore logarithmically concave and so

$$f(x)\,f(y)\leqslant\left(f\left(\frac{x+y}{2}\right)\right)^2,$$

i.e.

$$\left(1-\frac{1}{x}\right)^x\left(1-\frac{1}{y}\right)^y\leqslant\left(1-\frac{1}{\frac{x+y}{2}}\right)^{x+y},$$

which implies

$$\left(\frac{x-1}{x}\right)^x\left(\frac{y-1}{y}\right)^y\leqslant\left(\frac{\frac{x+y}{2}-1}{\frac{x+y}{2}}\right)^{x+y},$$

i.e.

$$\frac{(x-1)^x(y-1)^y}{\left(\frac{x+y}{2}-1\right)^{x+y}}\leqslant\frac{x^x y^y}{\left(\frac{x+y}{2}\right)^{x+y}}.$$

The proof is complete.

Remark. An extension of inequality (2) to n variables was also given in [2]. In fact it was proved that

$$\frac{\prod_{i=1}^{n}(x_i-1)^{p_i x_i-1}}{\left(\frac{\sum_{i=1}^{n}p_i x_i-\sum_{i=1}^{n}p_i}{\sum_{i=1}^{n}p_i}\right)^{\sum_{i=1}^{n}p_i x_i-\sum_{i=1}^{n}p_i}}\geqslant\frac{\prod_{i=1}^{n}\Gamma(x_i)^{p_i}}{\Gamma\left(\frac{\sum_{i=1}^{n}p_i x_i}{\sum_{i=1}^{n}p_i}\right)^{\sum_{i=1}^{n}p_i}}\geqslant\frac{\prod_{i=1}^{n}x_i^{p_i x_i}}{\left(\frac{\sum_{i=1}^{n}p_i x_i}{\sum_{i=1}^{n}p_i}\right)^{\sum_{i=1}^{n}p_i x_i}}$$

which for $n=2$, $p_1=p_2=1$ reduces to (2).

There are other inequalities concerning the ratio $\dfrac{\Gamma(x)\Gamma(y)}{\left(\Gamma\left(\frac{x+y}{2}\right)\right)^2}$. For example, in [3] it was proved that

$$\frac{(y-x)^2+4y}{4y}\leqslant\frac{\Gamma(x)\Gamma(y)}{\left(\Gamma\left(\frac{x+y}{2}\right)\right)^2},\tag{3}$$

while in [4] we find

$$1+\frac{(y-x)^2(y-2)}{(x+y-2)^2}\leqslant\frac{\Gamma(x)\Gamma(y)}{\left(\Gamma\left(\frac{x+y}{2}\right)\right)^2}.\tag{4}$$

Inequalities (3) and (4) are also cited in monograph [5]. However, these inequalities cannot be compared with (1) since, for $x=3$, $y=2$, we have

$$\frac{(y-x)^2+4y}{4y}>\frac{(x-1)^x(y-1)^y}{\left(\frac{x+y}{2}-1\right)^{x+y}}>1+\frac{(y-x)^2(y-2)}{(x+y-2)^2},$$

while for $x=2$, $y=4$ we get

$$1+\frac{(y-x)^2(y-2)}{(x+y-2)}>\frac{(x-1)^x(y-1)^y}{\left(\frac{x+y}{2}-1\right)^{x+y}}>\frac{(y-x)^2+4y}{4y}.$$

2. Some reasonably interesting inequalities involving the gamma function, or other special functions can be obtained by an application of the following elementary result.

If f is an integrable function and $f(t_1, \ldots, t_n) \leqslant M$ for $(t_1, \ldots, t_n) \in V \subset R^n$, then

$$\int_V \cdots \int f(t_1, \ldots t_n)\, dt_1 \cdots dt_n \leqslant M\, \mu(V),$$

where $\mu(V)$ is the volume of V.

We shall give three examples to illustrate this application:

Example 1. Start with the formula

$$\underset{\substack{t_i \geq 0 \\ t_1 + \cdots + t_n \leq 1}}{\int \cdots \int} t_1^{x_1 - 1} \cdots t_n^{x_n - 1}\, dt_1 \cdots dt_n = \frac{\Gamma(x_1) \cdots \Gamma(x_n)}{\Gamma(x_1 + \cdots + x_n + 1)},$$

which holds for $x_i > 0$ $(i = 1, \ldots, n)$.

Since

$$\max_{\substack{t_i \geq 0 \\ t_1 + \cdots + t_n \leq 1}} \left(t_1^{x_1 - 1} \cdots t_n^{x_n - 1}\right) = \frac{(x_1 - 1)^{x_1 - 1} \cdots (x_n - 1)^{x_n - 1}}{(x_1 + \cdots + x_n - n)^{x_1 + \cdots + x_n - 1}}$$

and

$$\underset{\substack{t_i \geq 0 \\ t_1 + \cdots + t_n \leq 1}}{\int \cdots \int} dt_1 \cdots dt_n = \frac{1}{n!},$$

we obtain the inequality

$$\frac{\Gamma(x_1) \cdots \Gamma(x_n)}{\Gamma(x_1 + \cdots + x_n + 1)} \leqslant \frac{(x_1 - 1)^{x_1 - 1} \cdots (x_n - 1)^{x_n - 1}}{n!\,(x_1 + \cdots + x_n - n)^{x_1 + \cdots + x_n - 1}}.$$

Example 2. The following formula

$$\int_V \cdots \int e^{a_1 t_1 + \cdots + a_n t_n}\, dt_1 \cdots dt_n = \pi^{n/2} \sum_{k=0}^{\infty} \frac{(a_1^2 + \cdots + a_n^2)^k}{2^{2k}\, k!\, \Gamma\left(1 + \frac{n}{2} + k\right)}$$

holds for $a_i > 0$ $(i = 1, \ldots, n)$, where $V = \{(t_1, \ldots, t_n) \mid t_i \geqslant 0 \text{ and } t_1^2 + \cdots + t_n^2 \leqslant 1\}$.

However, we have

$$\mu(V) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)},$$

and

$$\max_{(t_1, \ldots, t_n) \in V} e^{a_1 t_1 + \cdots + a_n t_n} = \exp\sqrt{a_1^2 + \cdots + a_n^2},$$

and so

$$\sum_{k=0}^{\infty} \frac{(a_1^2 + \cdots + a_n^2)^k}{2^{2k}\, k!\, \Gamma\left(1 + k + \frac{n}{2}\right)} \leqslant \pi^{n/2} \frac{\exp\sqrt{a_1^2 + \cdots + a_n^2}}{\Gamma\left(\frac{n}{2} + 1\right)}.$$

This inequality for $n=8m$ and $r-\sqrt{a_1^2+\cdots+a_n^2}$ becomes

$$\sum_{k=0}^{\infty}\frac{r^{2k}}{2^{2k}\,k!\,\Gamma(1+k+4m)}\leqslant\frac{e^r\,\pi^{4m}}{\Gamma(4m+1)},$$

which implies

$$J_{4m}(ir)\leqslant\frac{\pi^{4m}\,e^r}{2^{4m}\,\Gamma(4m+1)},$$

where J_ν is Bessel's function of the first order.

Example 3. Let $p, q, r, s>0$. Then

$$\mathrm{B}(p+r,\,q+s)=\int_0^1 x^{p+r-1}(1-x)^{q+s-1}\,dx\leqslant\max_{0\leq x\leq 1}x^r(1-x)^s\int_0^1 x^{p-1}(1-x)^{s-1}\,dx$$
$$=\left(\max_{0\leq x\leq 1}x^r(1-x)^s\right)\mathrm{B}(p,\,q),$$

where B is the beta function.

Therefore, since

$$\max_{0\leq x\leq 1}x^r(1-x)^s=\left(\frac{r}{r+s}\right)^r\left(\frac{s}{r+s}\right)^s,$$

we obtain

$$\mathrm{B}(p+r,\,q+s)\leqslant\left(\frac{r}{r+s}\right)^r\left(\frac{s}{r+s}\right)^s\mathrm{B}(p,\,q),$$

or, substituting $p=P-r,\ q=Q-s\ (P>r,\ Q>s)$,

$$\mathrm{B}(P,\,Q)\leqslant\left(\frac{r}{r+s}\right)^r\left(\frac{s}{r+s}\right)^s\mathrm{B}(P+r,\,Q+s). \tag{5}$$

Inequality (5) was proved in [6].

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