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ON THE RECONSTRUCTION OF COUNTABLE FORESTS

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Abstract. We present a brief history of the reconstruction conjecture. Although counterexamples to the infinite version are known, we provide a restricting hypothesis and prove that certain infinite forests are reconstructable. On the other hand, we show that extensions of the theorem are unlikely by providing counterexamples for the most plausible weakening of the hypothesis. To state the theorem, we define an infinite graph as *almost r-regular* if all but a finite number of points have degree r. Our result is that every almost r-regular forest is reconstructible.

1. *History.* Ulam's conjecture [9], as restated by one of us [4] asserts that the collection of point deleted subgraphs (PDS's) determine the graph. That is, given the PDS's, the graph can be "reconstructed". In spite of considerable attention, the conjecture remains unsolved. Results have been obtained, however, for the conjecture restricted to trees, First, Kelly [7] showed that all trees are reconstructable. Then Harary and Palmer [6] proved the stronger result that every tree can be reconstructed from its endpoint PDS's. Finally, Bondy [1] proved that only the peripheral PDS's are needed to reconstruct trees. And Manvel [8] observed that in general, the *set* of different PDS's determine a tree.

Harary [4] proposed the Reconstruction Conjecture for infinite graphs. This was shot down by Fisher [3], who found a pair of non-reconstructable infinite graphs, but his counterexample was not acyclic, nor locally finite, nor connected. Consequently, there was still hope for an infinite reconstruction theorem with a more restrictive hypothesis! Since acyclic finite graphs are reconstructable, possibly the conjecture held for infinite forests. But then we [2] found a countable collection of nonreconstructable countable forests. Consequently. we were led to further restrict the hypothesis to almost locally finite forests, that is, countable forests in which only finitely many points have infinite degree, and to formulate and prove a corresponding theorem.

2. *A class of reconstructable infinite forests.* We generally adhere to the terminology in the book [5], but infinite graphs will occur frequently. In addition, we say an infinite graph is *almost r-regular* if all but a finite number of points have degree r. The finitely many points of degree $\neq r$ are called *nonregular.* As a special case, every finite graph shall be considered to be almost r-regular for all non-negative integers r. Let T_r denote the unique tree that is regular of degree r. Obviously T_0 and T_1 are finite trees; the remaining T_r are infinite trees. For $r \geqslant 1$, let S_r be one of the r identical components formed by deleting a point from T_r.

It can be easily determined from the point deleted subgraphs whether G is an almost r-regular forest. Namely, G is an almost r-regular forest if and only if every G_i is.

Theorem. *If G is an almost r-regular forest, then G is reconstructable.*

Proof. If $r=0$ or 1, all but finitely many of the components of G_i are identically T_r. Consequently, G is easily reconstructed from one of the identical subgraphs obtained by deleting a point of some T_r. Thus, we may assume that $r \geqslant 2$.

Case 1. *G has an infinite number of components.*

Since there are only finitely many nonregular points, all but a finite number of components muts be copies of T_r. Let n_i be the number of components different from T_r in G_i. Choose n to be the largest value of n_i which occurs for infinitely many i. This value of n must have resulted either from the removal of a point of T_r to produce r copies of S_r, or from the removal of a point of degree larger than r from some component other than T_r. Since the latter possibility occurs only finitely often, while the first possibility always yields the same graph, we take any G_i with $n_i = n$ such that there are infinitely many point deleted subgraphs isomorphic to G_i. Then the joining of r of the copies of S_r in G_i to a new point to produce a tree T_r reconstructs G.

Case 2. *The number k of components is finite.*

Then all but finitely many G_i have $k+r-1$ components, and we discard these finitely many exceptions. Now if no infinite component in any remaining G_i contains two nonregular points, then all the infinite components of G are T_r, and the approach in Case 1 reconstructs G. Consequently, we may assume that some G_i has an infinite component containing two nonregular points, and discard all remaining G_i which do not have this property.

We now construct a possible candidate for the reconstructed graph G. Let $V_{i,1}, V_{i,2}, \ldots, V_{i,n_i}$ be the nonregular points of G_i. Fix the point $V_{i,j}$ and let $d_{i,j}$ be the minimum distance in G_i between this point and the other nonregular points. Then let m_i be the maximum finite value of $d_{i,j}$ as j varies. Because G_i has an infinite component with at least two nonregular points, we are assured that m_i is finite. Without loss of generality, assume $V_{i,1}$ and $V_{i,2}$ are the two points attaining m_i. With just finitely many exceptions, which we discard, one of $V_{i,1}$ or $V_{i,2}$ say $V_{i,1}$, has the property that every $V_{i,1}-V_{i,j}$ path contains $V_{i,2}$ and the degree of $V_{i,1}$ is $r-1$. Finally, we discard the finitely many G_i which do not contain at least $r-1$ copies of S_r. For the remaining G_i we form the graph G_i^* by inserting one new point of degree r, and making it adjacent to $V_{i,1}$ and the $r-1$ nonregular points of $(r-1)\,S_r$. This resulting graph is our candidate for G.

We may have reconstructed a wrong graph in finitely many cases. However, whenever G_i had been formed by the removal of a point of degree r which was sufficiently far out on one of the regular branches of some component, then our construction replaced the omitted point with just the right adjacencies to reproduce G. Here "sufficiently far out" refers to any point such that the m_i produced is strictly greater than the value of m for the original G. Curiously, we cannot at this point determine m in order to decide which m_i represent the removal of a point that is "sufficiently far out". However, we can observe that whatever m is, only finitely many G_i have $m_i \leqslant m$. Thus we are mistaken in the construction of only finitely many G_i^*. Therefore we discard the finitely many exceptional G_i^*'s and retain the infinitely many identical G_i^*'s. Any one of these G_i^*'s is the original graph G, completing the proof of the theorem.

3. *A collection of counterexamples.* There is no point in trying to extend the previous theorem to almost r-regular graphs of infinite degree r, because we [2] have already provided the counterexample of the $\aleph_0$-regular tree. Consequently, we can only extend the theorem by weakening the property of being almost r-regular. One possibility is to consider forests in which each component tree is almost r-regular, but then the whole forest fails to have this property because it has infinitely many components each containing finitely many nonregular points. The following construction provides a pair of nonreconstructable forests with this property for each choice of $r \geqslant 2$.

Let $\mathcal{C}_r = \{T : T$ is an almost r-regular tree$\}$. Note that $\mathcal{C}_r$ includes the set of all finite trees as a subset. We show that $\mathcal{C}_r$ is countable. The nonregular points of T induce a finite tree T^* consisting of these points and all paths joining them. Now T can be formed from T^* by adding copies of S_r with a new line from some point of T^* to the nonregular point of S_r. This increases the degree of one point of T^* by one, so the process is continued until the points from T^* have the same degree they had originally in T. Then the tree so formed is in fact identical to T. Since only a finite number of S_r's are added, and since the T^*'s are finite, the resulting number of possible trees, namely $|\mathcal{C}_r|$, is necessarily countable. The collection of trees $\mathcal{R}_r = \mathcal{C}_r - \{T_r\}$ is useful in constructing conterexamples. We form two graphs G_r and H_r as follows. Let G_r be the union of a countable number of copies of each tree in $\mathcal{R}_r$, and let H_r be the union of G_r with a single copy of T_r. Both G_r and H_r are countable graphs.

Now any PDS of G_r occurs countably often since each point is in a countable similarity class. Noting that T_r is point-symmetric, we observe that H_r also has each PDS occurring countably often. Consequently, it suffices to ignore multiplicities, and simply show that G_r and H_r have the same set of PDS's.

Now the deletion of a point from G_r leaves the union of G_r with a finite number of copies of trees from $\mathcal{C}_r$. Those from $\mathcal{R}_r$ are absorbed into G_r, so the PDS has the form $G_r \cup nT_r$ where n is a nonnegative integer. Furthermore, each nonnegative integer occurs for some point of G_r.

Similarly, deleting a point from the G_r portion of H_r yields a PDS of the form $G_r \cup nT_r$, only now $n \geqslant 1$. But fortunately, the deletion of a point from the T_r portion of H_r produces $G_r = G_r \cup 0\, T_r$. Thus, H_r has the same PDS's as G_r. Moreover, G_r and H_r are not isomorphic since H_r has a component isomorphic to T_r and G_r does not.

The preceding family of counterexamples was defined in a rather existential manner, and consequently is larger than necessary. The following variation, which results from an alternate set to $\mathcal{C}_r$, will be minimal in the sense that no proper subset of the new set $\mathcal{C}_r'$ can serve to provide another nonreconstructable pair of forests.

Let $J_{r,n}$ be the tree obtained by joining one point from each of n copies of T_r to a new point which then has degree n. For example, $J_{2,n}$ with $n \leqslant 4$. is depicted in Figure. 1.

We now define

$$\mathscr{C}_r' = \{T : T \text{ is an almost } r\text{-regular subtree of } J_{r,n} \text{ for some } n\}.$$

Evidently, $\mathscr{C}_r'$ is countable since it is an infinite subset of $\mathscr{C}_r$. Proceed to define R_r', G_r' and H_r' as in the previous counterexample. The argument given there applies here unchanged, establishing that the proper subset $\mathscr{C}_r'$ of the $\mathscr{C}_r$ also serves to construct a counterexample.

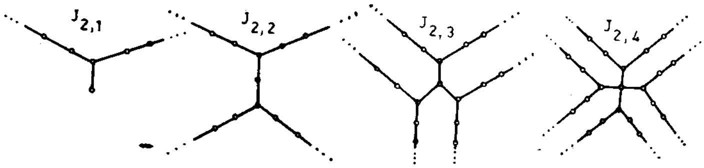

Figure 1. A Useful Family of Trees

We now show the minimality of $\mathscr{C}_r'$ for this purpose. Suppose $\mathscr{C}_r''$ is any set of trees used to define a pair of nonreconstructable forests in this way. In order to produce $G_r'' \cup nT_r$ as a point deleted subgraph, we must have $J_{r,n} \in \mathscr{C}_r''$. Moreover, any tree in $\mathscr{C}_r'$ formed by deleting finitely many points from some $J_{r,n}$, and consequently must be in $\mathscr{C}_r'$ since each successive component obtained by deleting one point at a time must be in $\mathscr{C}_r''$ in order that $G_r'' - v_i = G_r'' \cup nT_r$. Thus, we have shown $\mathscr{C}_r' \subset \mathscr{C}_r''$, so that $\mathscr{C}_r'$ is a minimal set which can be used to determine a counterexample in this way.

This collection of counterexamples leads us to believe that no extension to our theorem is likely. However, there is still an interesting open problem for infinite trees. Our counterexample [2] contains the regular tree of countable degree which is not reconstructable. Invoking local finiteness to forbid this one counterexample, we arrive at a statement which we believe to be true.

Conjecture: Every locally finite tree is reconstructable.

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