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THE CONVERSE TO AN ASCOLI'S TYPE THEOREM

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1. Introduction

In [4] the following theorem of Ascoli's type is proved: If X and Y are two arbitrary topological spaces and if $M(X, Y; \mathcal{K})$ denotes the space of all multi-valued functions on X to Y with the compact-open topology, then a closed set $\mathcal{F} \subset M(X, Y; \mathcal{K})$ is compact if at each point $x \in X$, $\mathcal{F}(x) = \bigcup \{F(x) \mid F \in \mathcal{F}\}$ has a compact closure in Y, and $\mathcal{F}$ is evenly continuous. The purpose of this paper is to prove the following converse:

Theorem. *Let X be a locally compact T_2 space, Y a regular T_2 space and $\mathcal{A} = \{F \in M(X, Y) \mid F$ is continuous and point-compact$\}$ have the compact-open topology. If a family $\mathcal{F} \subset \mathcal{A}$ is compact, then*

(a) *$\mathcal{F}$ is closed in $\mathcal{A}$*

(b) *$\mathcal{F}(x) = \bigcup \{F(x) \mid F \in \mathcal{F}\}$ has a compact closure in Y*

(c) *$\mathcal{F}$ is evenly continuous.*

2. Preliminaries

Let X and Y be two topological spaces. If $F(x)$ is a nonempty subset of Y for each $x \in X$, we say F is a multi-valued function or multifunction from X to Y and write $F: X \to Y$. If $F(x)$ is a non-empty closed (compact) subset of Y for each $x \in X$, we say that F is point-closed (point-compact). For any $A \subset X$, $F(A) = \bigcup \{F(x) \mid x \in A\}$ and for any $B \subset Y$, $F^{-1}(B) = \{x \in X \mid F(x) \cap B \neq \varnothing\}$. The complement of a set A will be denoted by $\complement A$ and the closure of A by $\overline{A}$.

Definition 2.1. A multifunction $F: X \to Y$ is continuous if for each open set V in Y, the set $F^{-1}(V)$ is open and the set $F^{-1}(Y - V)$ is closed in X.

Let $M(X, Y)$ be the set of all multifunctions on X to Y. For any $A \subset X$ and $B \subset Y$, let

$$(A, B) = \{F \in M(X, Y) \mid F(A) \subset B\}$$

$$)A, B(= \{F \in M(X, Y) \mid A \subset F^{-1}(B)\}$$

D e f i n i t i o n 2.2. [4] The *compact-open* topology $\mathcal{K}$ for $M(X, Y)$ is the topology defined, by taking the totality of (K, U) and $)L, V($ as sub-basic open sets, where K and L are any compact subsets of X and U and V are any open sets in Y.

Let $M(X, Y; \mathcal{K})$ denote the space of all multifunctions on X to Y with the compact-open topology.

D e f i n i t i o n 2.3. [4] A family $\mathcal{F} \subset M(X, Y)$ is *evenly continuous* if for each x in X, each y in Y, and each open neighborhood V of y there exists an open neighborhood U of x and an open neighborhood W of y such that:

$$\text{(a) } \mathcal{F} \cap)x, W(\subset)U, V(, \text{ and}$$

$$\text{(b) } \mathcal{F} \cap)x, W(\cap (x, V) \subset (U, V).$$

Let 2^Y be the set of all non-empty closed subsets of Y with the Vietoris topology in which a basic open set is of the form

$$\langle U_0 \rangle \cap \rangle U_1 \langle \cap \cdots \cap \rangle U_n \langle,$$

where U_i is open in Y for each $i = 0, 1, \ldots, n$. For an open U in Y

$$\langle U \rangle = \{A \in 2^Y \mid A \subset U\}, \quad \rangle U \langle = \{A \in 2^Y \mid A \cap U \neq \varnothing\}.$$

R e m a r k 2.4. If $X \to Y$ is point-closed, then F induces a singlevalued function $F^* : X \to 2^Y$ by $F^*(x) = F(x)$ and F is continuous in the sense of Definition 2.1 if and only if $F^* : X \to 2^Y$ is continuous [3].

3. Lemmas

L e m m a 3.1. *If A is an arbitrary subset of X and B is a closed subset of Y, then (A, B) and $)A, B($ are closed subsets of $M(X, Y; \mathcal{K})$.*

P r o o f. This follows immediately from[1)]

(1) $$\mathbf{C}(A, B) = \bigcup_{a \in A})a, \mathbf{C}B($$

(2) $$\mathbf{C})A, B(= \bigcup_{a \in A} (a, \mathbf{C}B).$$

Consider first the formula (1). We have

$$F \in (A, B) \Leftrightarrow \bigcup_{a \in A} F(a) \subset B \Leftrightarrow F(a) \subset B, \quad \forall a \in A \Leftrightarrow F \in (a, B),$$

$$\forall a \in A \Leftrightarrow F \in \bigcap_{a \in A} (a, B).$$

Hence,

(3) $$(A, B) = \bigcap_{a \in A} (a, B),$$

but

(4) $$\mathbf{C}(a, B) =)a, \mathbf{C}B(,$$

since $F \in \mathbf{C}(a, B) \Leftrightarrow F(a) \cap \mathbf{C}B \neq \varnothing \Leftrightarrow a \in F^{-1}(\mathbf{C}B) \Leftrightarrow F \in)a, \mathbf{C}B($ and so (3) and (4) implies (1).

[1)] Here, for example, (a, B) denote $(\{a\}, B)$.

On the other hand, since $F \in)A, B(\Leftrightarrow F(a) \cap B \neq \varnothing, \ \forall a \in A$ we have $F \in \mathbf{C})A, B(\Leftrightarrow \exists a \in A$ such that $F(a) \cap B = \varnothing \Leftrightarrow \exists a \in A$, so $F \in (a, \mathbf{C}B)$. This completes the proof of (2).

Lemma 3.2. *Let $\mathcal{D} = \{F \in M(X, Y) \mid F$ is point-closed$\}$ have the compact-open topology. If Y is a regular space, then $\mathcal{D}$ is a T_2 space.*

Proof. Let F and G be any two distinct multifunctions in $\mathcal{D}$; then there exists a point $x \in X$ such that $F(x) \neq G(x)$. Since Y is a regular space, 2^Y is a T_2-space [3]. Consequently, there exist open sets $U_0, U_1, \ldots, U_m$ and $V_0, V_1, \ldots, V_n$ in Y such that

(5) $$F(x) \in \langle U_0 \rangle \cap \rangle U_1 \langle \cap \cdots \cap \rangle U_m \langle ; \quad G(x) \in \langle V_0 \rangle \cap \rangle V_1 \langle \cap \cdots \cap \rangle V_n \langle$$

and

(6) $$\langle U_0 \rangle \cap \cdots \cap \rangle U_m \langle \cap \langle V_0 \rangle \cap \cdots \cap \rangle V_n \langle = \varnothing .$$

By (5),

(7) $$F(x) \subset U_0, \quad F(x) \cap U_i \neq \varnothing \quad (i = 1, \ldots, m)$$

$$G(x) \subset V_0, \quad G(x) \cap V_j \neq \varnothing \quad (j = 1, \ldots, n).$$

Hence, if

$$\mathcal{M} = (x, U_0) \cap)x, U_1(\cap \cdots \cap)x, U_m($$

$$\mathcal{N} = (x, V_0) \cap)x, V_1(\cap \cdots \cap)x, V_n($$

then $\mathcal{M} \cap \mathcal{D}$ and $\mathcal{N} \cap \mathcal{D}$ are open in $\mathcal{D}$, by (7) $F \in \mathcal{M} \cap \mathcal{D}$, $G \in \mathcal{N} \cap \mathcal{D}$ and by (6)

$$(\mathcal{M} \cap \mathcal{D}) \cap (\mathcal{N} \cap \mathcal{D}) = \varnothing .$$

Now define a multifunction $P : M(X, Y) \times X \to Y$ by taking

$$P(F, x) = F(x)$$

for every point (F, x) of $M(X, Y) \times X$. Then we have

Lemma 3.3. *A family $\mathcal{F} \subset M(X, Y)$ is evenly continuous if and only if for each x in X, each y in Y, and each open neighborhood V of y there exists an open neighborhood U of x and an open neighborhood W of y such that*

$$\text{(a)} \ [\mathcal{F} \cap)x, W(] \times U \subset P^{-1}(V)$$

$$\text{(b)} \ P([\mathcal{F} \cap)x, W(\cap (x, V)] \times U) \subset V.$$

Proof. This assertion follows easily from Definition 2.3.

Lemma 3.4. *If X is a locally compact T_2 space and $\mathcal{C} = \{F \in M(X, Y) \mid F$ is continuous$\}$ have the compact-open topology, then the multifunction*

$$Q : \mathcal{C} \times X \to Y,$$

where $Q = P \mid \mathcal{C} \times X$, is continuous.

Proof. Let V be an open set in Y. We will prove that $Q^{-1}(V)$ is open in $\mathcal{C} \times X$. Let

$$(F_0, x_0) \in Q^{-1}(V) \Leftrightarrow F_0(x_0) \cap V \neq \varnothing \Leftrightarrow x_0 \in F_0^{-1}(V).$$

Since $F_0 \in \mathscr{C}$ and V is open, $F_0^{-1}(V)$ is an open neighborhood of x_0. Consequently, since X is a locally compact T_2 space, there exists an open set U in X such that

$$x_0 \in U, \qquad \overline{U} \subset F_0^{-1}(V)$$

and $\overline{U}$ is compact and, hence, obviously

$$(F_0, x_0) \in [\mathscr{C} \cap)\overline{U}, V(] \times U \subset Q^{-1}(V).$$

It remains to prove that $Q^{-1}(Y-V)$ is closed. Let

(8) $\qquad (F_0, x_0) \in \mathbf{C} Q^{-1}(Y-V) \Leftrightarrow F_0(x_0) \cap (Y-V) = \varnothing \Leftrightarrow F_0(x_0) \subset V.$

Since $F_0 \in \mathscr{C}$ and V is open

$$W = \{x \in X \mid F_0(x) \subset V\} = X - F_0^{-1}(Y-V)$$

is an open set in X and, by (8), $x_0 \in W$. Thus, since X is a locally compact T_2 space, there exists an open set U in X such that

$$x_0 \in U, \qquad \overline{U} \subset W \qquad \text{and } \overline{U} \text{ is compact}$$

and, hence, obviously

$$(F_0, x_0) \in [\mathscr{C} \cap (U, V)] \times U \subset \mathbf{C} Q^{-1}(Y-V)$$

which completes the proof.

4. Proof of the Theorem

(a) By Lemma 3.2, the regularity of Y implies that $\mathscr{D}$ is a T_2 space. Since Y is a T_2 space $\mathscr{A}$ is, clearly, a subspace of $\mathscr{D}$, and so a T_2 space itself. Therefore, since $\mathscr{F} \subset \mathscr{A}$ is compact and $\mathscr{A}$ is T_2, $\mathscr{F}$ is closed in $\mathscr{A}$.

(b) For each point $x \in X$, the multifunction

$$P_x : \mathscr{A} \to Y$$

defined by $P_x(F) = F(x)$ is continuous, since for any given open set V in Y, we have

$$P_x^{-1}(V) = \{F \in \mathscr{A} \mid F(x) \cap V \neq \varnothing\} =)x, V(\cap \mathscr{A}$$

$$\mathbf{C} P_x^{-1}(Y-V) = \{F \in \mathscr{A} \mid F(x) \subset V\} = (x, V) \cap \mathscr{A}.$$

According to Remark 2.4 P_x induces a single-valued continuous function $P_x^* : \mathscr{A} \to 2^Y$ by $P_x^*(F) = P_x(F)$, and since $\mathscr{F}$ is compact so is

$$P_x^*(\mathscr{F}) = \{P_x^*(F) \mid F \in \mathscr{F}\} = \{F(x) \mid F \in \mathscr{F}\} \text{ in } 2^Y.$$

Thus, by [5]

$$\cup \{F(x) \mid F \in \mathscr{F}\} = \mathscr{F}(x)$$

is compact in Y. Since Y is a regular space, the compact set $\mathscr{F}(x)$ has a compact closure in Y.

(c) Let $x \in X$, $y \in Y$ and an open set V containing y be arbitrarily given. Since Y is a regular space, there exists an open neighborhood W of y, such that $\overline{W} \subset V$. By Lemma 3.1, $)x, \overline{W}($ is closed in $M(X, Y; \mathscr{K})$ and, since $\mathscr{F}$ is compact, $K = \mathscr{F} \cap)x, \overline{W}($ is, obviously, a compact set in $\mathscr{A}$. It follows that $K \times \{x\}$ is compact in $\mathcal{A} \times X$. Moreover, because $\overline{W} \subset V$, it is easily to see that

$$K \times \{x\} \subset R^{-1}(V)$$

where $R = Q \mid \mathcal{A} \times X$, is continuous (Lemma 3.4.). Hence, since R is continuous and V is open, $R^{-1}(V)$ is an open set in $\mathcal{A} \times X$ containing a compact set $K \times \{x\}$. Therefore, using Wallace's theorem [2], it follows that there exists an open neighborhood U_1 of x such that

$$K \times U_1 \subset R^{-1}(V).$$

Similarly, we can conclude that

$$L = \mathcal{F} \cap)x, \overline{W}(\cap (x, \overline{W})$$

is compact in $\mathcal{A}$; $L \times \{x\}$ is compact in $\mathcal{A} \times X$ and that[1)]

$$L \times \{x\} \subset (\mathcal{A} \times X) - R^{-1}(Y - V) \equiv E$$

where E is open in $\mathcal{A} \times X$. Consequently, there exists an open neighborhood U_2 of x, such that $L \times U_2 \subset E$. Let $U = U_1 \cap U_2$. Then, we have

$$\text{(9)} \qquad [\mathcal{F} \cap)x, W(] \times U \subset [\mathcal{F} \cap)x, \overline{W}(] \times U_1 = K \times U_1 \subset R^{-1}(V) \subset P^{-1}(V)$$

and

$$\text{(10)} \qquad [\mathcal{F} \cap)x, W(\cap (x, W)] \times U \subset [\mathcal{F} \cap)x, \overline{W}(\cap (x, \overline{W})] \times U_2 = L \times U_2 \subset E.$$

It follows from (10) that

$$\text{(11)} \quad P\big([\mathcal{F} \cap)x, W(\cap (x, W)] \times U\big) = R\big([\mathcal{F} \cap)x, W(\cap (x, W)] \times U\big) \subset R(E) \subset V.$$

Hence, the proof is completed by (9), (11) and Lemma 3.3.

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1) $(F, x) \in L \times \{x\} \Rightarrow F(x) \subset V$ since $\overline{W} \subset V$; on the other hand $(F, x) \in E \Leftrightarrow R(F, x) \subset \subset V \Leftrightarrow F(x) \subset V$.

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