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FIXED POINT THEOREMS FOR MAPPINGS WITH A GENERALIZED CONTRACTIVE ITERATE AT A POINT

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1. Introduction. Let (M, d) be a metric space and $T: M \to M$ a mapping. A mapping T is called a contraction if there is a positive number $q<1$ such that

$$d(Tx, Ty) \leqslant qd(x, y) \tag{1}$$

holds for all $x, y \in M$. A mapping T is said to be with a contractive iterate at a point $x \in M$ if there is a positive integer $n(x)$ such that for all $y \in M$

$$d(T^{n(x)} x, \quad T^{n(x)} y) \leqslant qd(x, y). \tag{2}$$

V. Sehgal [4] proved the following result:

Theorem 1. *Let (M, d) be a complete metric space and $T: M \to M$ a continuous mapping with a contractive iterate at each point $x \in M$. Then T has a unique fixed point $u \in M$ and $T^n x \to u$ for each $x \in M$.*

The result of Sehgal is generalized by L. Guseman [3] to mappings which are not necessarily continuous and which have a contractive iterate at each point in a subset of the space. Guseman proved the following results:

Theorem 2. *Let (M, d) be a complete metric space and let $T: M \to M$ be a mapping. Suppose there exists $B \subset M$ such that*

(a) *$TB \subset B$,*

(b) *for some $q<1$ and each $y \in B$ there is an integer $n(y) \geqslant 1$ with*

$$d(T^{n(y)} x, \quad T^{n(y)} y) \leqslant qd(x, y)$$

for all $x \in B$, and

(c) *for some $x_0 \in B$, $cl\{T^n x_0 : n \geqslant 1\} \subset B$.*

Then there is a unique $u \in B$ such that $Tu = u$ and $T^n z \to u$ for each $z \in B$. Furthermore, if $d(T^{n(u)} x, T^{n(u)} u) \leqslant q \cdot d(x, u)$ for all $x \in M$, then u is unique in M and $T^n x \to u$ for each $x \in M$.

Theorem 3. *Let (M, d) be a metric space, $T: M \to M$ a mapping, and $u, x_0 \in M$ with $T^n x_0 \to u$. If T is with a contractive iterate at a point u, then T has a unique fixed point u and $T^n y \to u$ for each $y \in M$.*

In [1] we considered generalized contractions, defined as follows:

A mapping $T: M \to M$ is said to be a generalized contraction iff for every $x, y \in M$ there are non-negative numbers $q(x, y)$, $r(x, y)$, $s(x, y)$, and $t(x, y)$ with $\sup_{x, y \in M} \{q(x, y) + r(x, y) + s(x, y) + 2t(x, y)\} = k < 1$ and such that

$$(3) \quad d(Tx, Ty) \leq q(x, y)\, d(x, y) + r(x, y)\, d(x, Tx) + s(x, y)\, d(y, Ty) + \\ + t(x, y)\,(d(x, Ty) + d(y, Tx)).$$

It is clear that T is a generalized contraction if and only if T satisfies the following condition

$$(4) \quad d(Tx, Ty) \leq q \max \left\{ d(x, y);\quad d(x, Tx);\quad d(y, Ty);\quad \frac{1}{2}(d(x, Ty) + d(y, Tx)) \right\}$$

for all $x, y \in M$ and for some positive number $q < 1$. In [3] we proved a fixed point theorem for mappings which satisfy the condition

$$d(Tx, Ty) \leq q \max \{d(x, y);\quad d(x, Tx);\quad d(y, Ty);\quad d(x, Ty);\quad d(y, Tx)\}.$$

In the present paper we investigate mappings which are not necessarily continuous and satisfy the condition: there exists a $q < 1$ such that for each $x \in B \subset M$ there is a positive integer $n(x)$ such that for all $y \in B$

$$(5) \quad d(T^{n(x)} x, T^{n(x)} y) \leq q \max \left\{ d(x, y);\quad \frac{1}{2}(d(x, T^{n(x)} x) + d(y, T^{n(x)} y));\quad d(x, T^{n(x)} y);\quad d(y, T^{n(x)} x) \right\}.$$

Also we investigate mappings which satisfy a somewhat stronger condition

$$(6) \quad d(T^{n(x)} x, T^{n(x)} y) \leq q \max \left\{ d(x, y);\quad \frac{1}{3}(d(x, T^{n(x)} x) + d(y, T^{n(x)} y));\quad \frac{1}{3}(d(x, T^{n(x)} y) + d(y, T^{n(x)} x)) \right\}.$$

We present results which extend some of the results concerning generalized contractions of [1] to maps which satisfy (5) or (6), and also generalize the results of Guseman [3] and Sehgal [4].

2. In the following M will denote a metric space with a metric d. A mapping $T: M \to M$ is called orbitally continuous iff $u = \lim_i T^{n_i} x_0$ implies $Tu = \lim_i TT^{n_i} x_0$. The space M is called T-orbitally complete, where T maps M into M, iff every Cauchy sequence of the form $\{T^{n_i} x : i \in N\}$ converges in M.

The following is a generalization of the theorem 3:

Theorem 4. *Let $T: M \to M$ be a mapping and $u, x_0 \in M$ with $u = \lim_n T^n x$. If T satisfies (5) at u, i.e., if there are a positive integer $n(u)$ and a positive number $q < 1$ such that*

$$d(T^{n(u)} x, T^{n(u)} u) \leqslant q \max \left\{ d(x, u);\ \frac{1}{2}(d(x, T^{n(u)} x) + d(u, T^{n(u)} u));\ d(x, T^{n(u)} u);\ d(u, T^{n(u)} x)) \right\} \tag{7}$$

holds for all $x \in M$, then T has a unique fixed point u in M and $\lim_n T^n x = u$ for each $x \in M$.

Proof. From $\lim_n T^n x_0 = u$ it follows $\lim_n T^{n(u)} T^n x_0 = \lim_n T^{n+n(u)} x_0 = u$. Since T satisfies (7) we have

$$d(u, T^{n(u)} u) \leqslant d(u, T^n x) + d(T^n x, T^{n+n(u)} x) + d(T^{n(u)} T^n x, T^{n(u)} u) \leqslant$$
$$d(u, T^n x) + d(T^n x, T^{n+n(u)} x) + q \cdot a(T^n x, u),$$

where

$$a(T^n x, u) = \max \left\{ d(T^n x, u);\ \frac{1}{2}(d(T^n x, T^{n+n(u)} x) + d(u, T^{n(u)} u));\ d(T^n x, T^{n(u)} u);\ d(u, T^{n+n(u)} x) \right\}.$$

Hence

$$d(u, T^{n(u)} u) \leqslant d(u, T^n x) + d(T^n x, T^{n+n(u)} x) + q \cdot d(T^n x, u)$$

if $a(T^n x, u) = d(T^n x, u)$, or

$$d(u, T^{n(u)} u) \leqslant \frac{1}{2-q}(2\, d(u, T^n x) + (2+q)\, d(T^n x, T^{n+n(u)} x))$$

if $a(T^n x, u) = \frac{1}{2}(d(T^n x, T^{n+n(u)} x) + d(u, T^{n(u)} u)$, or

$$d(u, T^{n(u)} u) \leqslant \frac{1}{1-q}(d(u, T^n x) + d(T^n x, T^{n+n(u)} x) + q \cdot d(T^n x, u))$$

if $a(T^n x, u) = d(T^n x, T^{n(u)} u)$, or

$$d(u, T^{n(u)} u) \leqslant d(u, T^n x) + d(T^n x, T^{n+n(u)} x) + q \cdot d(u, T^{n+n(u)} x)$$

if $a(T^n x, u) = d(u, T^{n+n(u)} x)$.

Since $\lim_n T^n x = u$, it follows that $d(u, T^{n(u)} u) = 0$, i.e. $T^{n(u)} u = u$. By (7) u is the unique fixed point for $T^{n(u)}$ in M. Then $Tu = TT^{n(u)} u = T^{n(u)} Tu$ implies $Tu = u$. It is clear that u is the unique fixed point under T.

Now we shall show that $\lim_n T^n y = u$ for all $y \in M$. Let $b(y) = \max\{d(y, T^k y) : k = 1, 2, \ldots, n(u)\}$. If n is a positive integer then there are integers $r \geqslant 0$ and $0 \leqslant s \leqslant n(u) - 1$ such that $n = rn(u) + s$, and we have

$$d(u, T^n y) = d(u, T^{rn(u)+s} y) = d(T^{n(u)} u, T^{n(u)} T^{(r-1)n(u)+s} y) \leqslant$$

$$q \cdot \max \left\{ d(u, T^{(r-1)n(u)+s} y);\ \frac{1}{2} d(T^{(r-1)n(u)+s} y, T^n y); \right.$$

$$\left. d(u, T^n y);\quad d(T^{(r-1)n(u)+s} y, u) \right\}.$$

Since $d(u, T^n y) \leqslant q \cdot d(u, T^n y)$ is impossible, we have

$$d(u, T^{rn(u)+s} y) \leqslant q \cdot d(u, T^{(r-1)n(u)+s} y),$$

or

$$d(u, T^{rn(u)+s} y) \leqslant \frac{q}{2-q} d(u, T^{(r-1)n(u)+s} y) < q \cdot d(u, T^{(r-1)n(u)+s} y).$$

Therefore,

$$d(u, T^{rn(u)+s} y) \leqslant q \cdot d(u, T^{(r-1)n(u)+s} y).$$

Repeating this argument r times, we get

$$d(u, T^{rn(u)+s} y) \leqslant q^r \cdot d(u, T^s y) \leqslant q^r \cdot b(y).$$

But $n \to \infty$ implies $r \to \infty$, so $\lim_n d(u, T^n y) = 0$, and the proof is complete.

The proof of the following lemma is essentially the same as the proof of lemma in [4]:

Lemma. *If $T: M \to M$ is any mapping satisfying the condition* (5), *then for each $x \in B$, $r(x) = \sup_n d(x, T^n x)$ is finite.*

Proof. Let $x \in B$ and let $b(x) = \max\{d(x, T^k x) : k = 1, 2, \ldots, n(x)\}$. For a given positive integer n, let $r \geqslant 0$ and $0 \leqslant s \leqslant n(x) - 1$ be such that $n = rn(x) + s$. Then from

$$d(x, T^{rn(x)+s} x) \leqslant d(x, T^{n(x)} x) + d(T^{n(x)} x, T^{n(x)} T^{(r-1)n(x)+s} x) \leqslant$$

$$d(x, T^{n(x)} x) + q \cdot c(x, T^{(r-1)n(x)+s} x),$$

with $c = c(x, T^{(r-1)n(x)+s} x) = \max \Big\{ d(x, T^{(r-1)n(x)+s} x);$

$\frac{1}{2}(d(x, T^{n(x)} x) + d(T^{(r-1)n(x)+s} x, T^n x));\quad d(x, T^n x);\quad d(T^{(r-1)n(x)+s} x, T^{n(x)} x) \Big\}$,

it follows

(i) $$d(x, T^{rn(x)+s} x) \leqslant d(x, T^{n(x)} x) + q \cdot d(x, T^{(r-1)n(x)+s} x)$$

if $c = d(x, T^{(r-1)n(x)+s} x)$, or

(ii) $$d(x, T^{rn(x)+s} x) \leqslant \frac{2+q}{2-q} (d(x, Tx) + \frac{q}{2-q} d(x, T^{(r-1)n(x)+s} x)$$

if $c=\frac{1}{2}(d(x, T^{n(x)}x)+d(T^{(r-1)n(x)+s}x, T^n x))$, or

(iii) $$d(x, T^{rn(x)+s}x)\leqslant\frac{1}{1-q}d(x, T^{n(x)}x)$$

if $c=d(x, T^n x)$, or

(iiii) $$d(x, T^{rn(x)+s}x)\leqslant(1+q)d(x, T^{n(x)}x)+q\cdot d(x, T^{(r-1)n(x)+s}x)$$

if $c=d(T^{(r-1)n(x)+s}x, T^{n(x)}x)$. Since $\max\left\{1, \frac{2+q}{2-q}, \frac{1}{1-q}, (1+q)\right\}=\frac{1}{1-q}$ and $\max\left\{q, \frac{q}{2-q}, 0\right\}=q$, in any case of (i), (ii), (iii), (iiii) we have

$$d(x, T^{rn(x)+s}x)\leqslant\frac{1}{1-q}d(x, T^{n(x)}x)+q\cdot d(x, T^{(r-1)n(x)+s}x).$$

Repeating this argument r-times we get

$$d(x, T^{rn(x)+s}x)\leqslant\frac{1}{1-q}d(x, T^{n(x)}x)+q\frac{1}{1-q}d(x, T^{n(x)}x)+\cdots+$$
$$+q^{r-1}\frac{1}{1-q}d(x, T^{n(x)}x)+q^r d(x, T^s x)\leqslant$$
$$(1+q+\cdots+q^{r-1})\frac{1}{1-q}b(x)+q^r\frac{1}{1-q}b(x)\leqslant\left(\frac{1}{1-q}\right)^2\cdot b(x).$$

Since $n=rn(x)+s$ was arbitrary, we proved that

$$d(x, T^n x)\leqslant\left(\frac{1}{1-q}\right)^2\cdot\max\{d(x, T^k x): k=1, 2, \ldots, n(x)\}\cdot$$

Hence $r(x)=\sup_n d(x, T^n x)$ is finite.

Now we can prove the following result:

Theorem 5. *Let $T: M\to M$ be a mapping of a T-orbitally complete metric space M into itself. Suppose there exists $x_0\in M$ with $\overline{0}(x_0)=cl\{T^n x_0: n\in N\}$ such that for some $q<1$ and each $x\in\overline{0}(x_0)$ there is a positive integer $n(x)$ such that T satisfies the condition* (6) *for all $y\in\overline{0}(x_0)$. Then $\lim_n T^n x_0=u$ and $Tu=u$. Furthermore, if T satisfies* (7) *at u, then u is the unique fixed point in M and $\lim_n T^n x=u$ for each $x\in M$.*

Proof. It is clear that if T satisfies (6), then T also satisfies (5) on $\overline{0}(x_0)$. Therefore $r(x_0)=\sup_n d(x_0, T^n x_0)$ is finite. Consider the sequence

$$x_0, x_1=T^{n(x_0)}x_0, x_2=T^{n(x_1)}x_1, \ldots, x_{i+1}=T^{n(x_i)}x_i, \ldots.$$

For an arbitrary positive integer s we have

$$d(x_i, T^s x_i)=d(T^{n(x_{i-1})}x_{i-1}, T^{n(x_{i-1})}T^s x_{i-1})\leqslant$$

$$q\cdot\max\left\{d(x_{i-1}, T^s x_{i-1});\quad \frac{1}{3}(d(x_{i-1}, T^{n(x_{i-1})}x_{i-1})+d(T^s x_{i-1}, T^s x_i));\quad \frac{1}{3}(d(x_{i-1}, T^s x_i)+d(T^s x_{i-1}, x_i))\right\}.$$

Hence

$$d(x_i, T^s x_i)\leqslant q\cdot d(x_{i-1}, T^s x_{i-1}),$$

or

$$d(x_i, T^s x_i)\leqslant q\cdot\max\{d(x_{i-1}, T^{n(x_{i-1})}x_{i-1});\quad d(x_{i-1}, T^s x_{i-1});\quad d(x_{i-1}\, T^{s+n(x_{i-1})}x_{i-1})\},$$

or

$$d(x_i, T^s x_i)\leqslant q\cdot\max\{d(x_{i-1}, T^s x_i);\quad d(x_{i-1}, T^s x_{i-1});\quad d(x_{i-1}, T^{n(x_{i-1})}x_{i-1})\}.$$

Therefore,

$$d(x_i, T^s x_i)\leqslant q\cdot\max\{d(x_{i-1}, T^s x_{i-1});\quad d(x_{i-1}, T^{n(x_{i-1})}x_{i-1});\quad d(x_{i-1}, T^{s+n(x_{i-1})}x_{i-1})\}.$$

Repeating this argument, one has

$$d(x_i, T^s x_i)\leqslant q^2\cdot\max\{d\, x_{i-2}, T^s x_{i-2});\quad d(x_{i-2}\, T^{n(x_{i-2})}x_{i-2});\quad d(x_{i-2}, T^{s+n(x_{i-2})}x_{i-2});\quad d(x_{i-2}, T^{n(x_{i-1})}x_{i-2});\quad d(x_{i-2}, T^{n(x_{i-1})+n(x_{i-2})}x_{i-2});\quad d(x_{i-2}, T^{s+n(x_{i-1})}x_{i-2});\quad d(x_{i-2}, T^{s+n(x_{i-1})+n(x_{i-2})}x_{i-2})\leqslant\cdots\leqslant q^i r(x_0).$$

Hence, it follows that the sequence $\{T^n x_0 : n\in N\}$ is Cauchy. Since M is T-orbitally complete, there is $u\in\overline{0}(x_0)$ such that

$$u=\lim_n T^n x_0.$$

Since the condition (6) implies the condition (5), we have $Tu=u$ by the theorem 4. The last assertion of the theorem follows directly from the theorem 1. This completes the proof.

REFERENCES

[1] Lj. Ćirić, *Generalized contractions and fixed-point theorems*, Publications Inst. Math., Nouv. série, tome 12 (26), 1971, 19—26.

[2] Lj. Ćirić, *On contraction type mappings*, Mathematica Balkanica 1 (1971), 52—57.

[3] Lj. Ćirić, *An extension of Banach's theorem on contraction mappings*, (to appear).

[4] L. Guseman, *Fixed point theorems for mappings with a contractive iterate at a point*, Proc. Amer. Math. Soc. 26 (1970), 615—618.

[5] V. Sehgal, *A fixed point theorem for mappings with a contractive iterate*, Proc. Amer. Math. Soc. 23 (1969), 631—634.