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ON THE EXPONENTIAL PROPERTIES OE THE IMPLICATION

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In the mathematical logic, in the chapter on Boolean functions, one uses the exponential notation for the components in the disjunctive normal form of the function. As known, the function $f(x_1, \ldots, x_n)$ whose arguments and values belong to the set (0, 1), is given by the identity

$$f(x_1, \ldots, x_n) = \sum_{(y_1, \ldots, y_n)} f(y_1, \ldots, y_n)\, x_1^{y_1} \ldots x_n^{y_n} \tag{1}$$

where the summation, in fact the antivalence, should be taken for all choices of $y_1, \ldots, y_n \in (0, 1)$ and x^y is explained as x' for $y=0$, x for $y=1$. Instead of the antivalence one could put the disjunction in the formula (1) because for each choice of $y_1, \ldots, y_n$ only one term is equal to 1. The exponential function introduced in this way is by no means a real exponential function. As one could easily check this function has no other use in the current mathematical literature. (cf. [2], [3], [4]).

In fact, this function is a logical equivalence as one could check by the truth value table. This reveals the pleonasmic character of the formula (1). The formula tells us that the function $f(x_1, \ldots, x_n)$ is equal to $f(y_1, \ldots, y_n)$ if and only if $x_1 = y_1, \ldots, x_n = y_n$. This is evident if we write correctly $x = y$ instead of x^y:

$$f(x_1, \ldots, x_n) = \sum_{(y_1, \ldots, y_n)} f(y_1, \ldots, y_n)\,(x_1 = y_1) \cdots (x_n = y_n). \tag{2}$$

In this article we shall investigate the real exponential function in the mathematical logic and describe its relation to other logical connectives. We start from our aim to find the function which has to have the following properties

(i) $$(x^y)^z = x^{yz},$$

(ii) $$x^y x^z = x^{y \vee z},$$

(iii) $$x^y z^y = (xz)^y,$$

where xy is the conjuction of x, y, the disjunction being $x \vee y$. Withre spect to the conjunction $\cdot$ and antivalence $+$ we have the two-element field B_2. We treat the quoted properties as functional equations over this field. Each function $f(x, y)$ of two variables x, y must have the form

$$a+bx+cy+dxy, \quad a, b, c, d \in 0, 1.$$

The property (i) reduces to

$$f(f(x, y), z)=f(x, yz).$$

The left side of the last equation having the form

$$(x^y)^z = a+(a+bx+cy+dxy)\,b+cz+d\,(a+bx+cy+dxy)\,z$$

and the left side

$$x^{yz} = a+bx+cyz+dxyz,$$

we must have the system of Boolean equations

$$(3) \qquad \begin{aligned} ab=bc=cd=bd=0,\\ c+ad=0. \end{aligned}$$

The only solutions of these equations are

$$a=b=c=d=0,$$

$$a=1,\ b=c=d=0,$$

$$b=1,\ a=c=d=0,$$

$$a=c=d=1,\ b=0,$$

$$a=b=c=0,\ d=1,$$

giving respectively the functions

$$0,\ 1,\ x \quad y \to x,\ xy.$$

The first three functions are trivial ones, the last, xy, does not have the property (ii) as one could see from the case $x=z=1$, $y=0$.

For the function xy the property (i) reduces merely to the associativity law for the conjuction. It remains only the forth function $y \to x$ or $1+y+xy$, or in the exponential form x^y. So we have the

Theorem. *The functional equation*

$$(x^y)^z = x^{yz}$$

over the field B_2 *has the solutions* $0, 1, x, 1+y+xy, xy$. *From these the only non-trivial solution (that is, the solution depending essentially on both variables* x, y*) satisfying also the functional equations* (i), (ii) *is the function* $1+y+xy$.

Definition 1. — From now on x^y means only $y \to x$ (or equivalently $1+y+xy$).

It follows that $0^x = x'$ contrary to the notation x^0 adopted in the literature. (f.e. [4] p. 10 — Yablonsky, Gawriloff, Kudryavceff).

By defining this function we have the possibility to check out many of the known tautologies. For example: $z \to (y \to x) = yz \to x$ is just the relation

$$(x^y)^z = x^{yz}.$$

The tautology $(y \to x)(z \to x) = (y \vee z) \to x$ is only the new way to write $x^y x^z = x^{y \vee z}$, while $(y \to x)(y \to z) = y \to xz$ represent the fact

$$x^y z^y = (xz)^y.$$

It is easy to prove the following

Corollary 1.

$$(x+y)^z = x^z + yz \qquad \text{(binomial formula)},$$

$$\frac{x,\ y^x}{y} \qquad \text{(modus ponens)}$$

$$(x \vee y)^z = x^z + x^{yz} + 1; \quad x^y + y^x = (x+y)';$$

$$x^{(y^z)} = z^x + y^z + xyz.$$

The tautology $y \to (x \vee y)$ is just the identity $(x \vee y)^z = x^z + x^{yz} + 1$ from our corollary for $y = z$.

Similarly $(x \to y) \to x = x$ reduces to the identity $x^{(y^z)} = z^x + y^z + xyz$ for $x = z$, and so on.

In order to illustrate the practical use of the function x^y we shall solve the problem of the interpolation for Boolean functions.

Theorem. 2. — *Every Boolean function $f(x_1, \dots, x_n)$ can be represented in the form*

$$f(x_1, \dots, x_n) = \sum_{(Y)} c_{y_1 \dots y_n} x_1^{y_1} \cdots x_n^{y_n}, \qquad Y = (y_1, \dots, y_n) \tag{3}$$

where

$$c_{y_1 \dots y_n} = \sum_{(Z)} f(z_1, \dots, z_n), \qquad Z = (z_1, \dots, z_n),$$

for each solution of the equation

$$y_1^{z_1} \cdots y_n^{z_n} = 1$$

in the unknowns $z_1, \dots, z_n \in 0.1$.

Proof. — Let F be the vector whose coordinates are $f(y_1, \dots, y_n)$ listed in the lexicographical order from $f(0, \dots, 0)$ to $f(1, \dots, 1)$. Let C be the vector of unknown quantities $c_{y_1 \dots y_n}$ again in the lexicographical order and let W be the matrix whose rows $w_{z_1 \dots z_n}$ are ordered in the lexicographical order with respect to z and the vector $w_{z_1 \dots z_n}$ has the coordinates $z_1^{y_1} \cdots z_n^{y_n}$ in the lexicographical order with respect to y. With these conventions the system of linear algebraic equations (3) over the field B_2 is given by the condensed expression

$$F = WC$$

In the article [1] I have proved that W is a regular matrix and that $W^{-1}=W$. Therefore

$$C=W^{-1}F=WF.$$

From this we conclude that

$$c_{y_1\ldots y_n}=w_{z_1\ldots z_n}F=[z_1^0\cdots z_n^0,\ldots,z_1^1\cdots z_n^1][f(0,\ldots,0),\ldots,f(1,\ldots,1)]'$$
$$=\sum_{(Z)}f(z_1,\ldots,z_n),\ y_1^{z_1}\cdots y_n^{z_n}=1$$

as it was to be proved.

Thus for the free member of the polynomial reperesenting the function $f(x_1,\ldots,x_n)$ we have

$$c_{y_1\ldots y_n}=c_{0\ldots 0}=\sum_{(Z)}f(z_1,\ldots,z_n)=f(0,\ldots,0),\ \ y_1^{z_1}\cdots y_n^{z_n}=1$$

because the equation $0^{z_1}\cdots 0^{z_n}=1$ has as the unique solution $z_1=\cdots=z_n=0$. Similarly the coefficient $c_{11\ldots 1}$ of the member $x_1\cdots x_n$ is the antivalence of all values of the function f because the equation $1^{z_1}\cdots 1^{z_n}=1$ has the solution Z free of any restriction.

Quite an elegant way of finding the vector C could be obtained by starting with the disjunctive normal form but now in the correct use of exponential function which gives

$$f(x_1,\ldots,x_n)=\sum_{(Y)}f(y_1,\ldots,y_n)\,x_1^{y_1}\cdots x_n^{y_n}y_1^{x_1}\cdots y_n^{x_n}.$$

The polynomial in $x_1,\ldots,x_n$

$$x_1^{y_1}\cdots x_n^{y_n}y_1^{x_1}\cdots y_n^{x_n}$$

for given choice of $y_1,\ldots,y_n\in 0,1$ is a polynomial of the Lagrangean type: it has the value 1 only once when $x_1=y_1,\ldots,x_n=y_n$. (Really, if for some $k=1,\ldots,n$ we have $x_k\neq y_k$ then in both cases $x_k=0,\ y_k=1$ or $x_k=1$, $y_k=0$, we would have $x_k^{y_k}y_k^{x_k}=0$ and the corresponding member disappears from the formula. Denote this polynomial with $L_{y_1\ldots y_n}$. It holds

$$L_{y_1\ldots y_n}=(1+x_1+y_1)\cdots(1+x_n+y_n)= \tag{4}$$
$$=\sum_{(Z)}x_1^{z_1}\cdots x_n^{z_n};\ z_1^{y_1}\cdots z_n^{y_n}=1.$$

It remains only to change the order of antivalences in the expression

$$f(x_1,\ldots,x_n)=\sum_{(Y)}f(y_1,\ldots,y_n)\sum_{(Z)}x_1^{z_1}\cdots x_n^{z_n},\ \ z_1^{y_1}\cdots z_n^{y_n}=1. \tag{5}$$

By doing this we get the desired result

$$f(x_1,\ldots,x_n)=\sum_{(Z)}x_1^{z_1}\cdots x_n^{z_n}\sum_{(Y)}f(y_1,\ldots,y_n),\ z_1^{y_1}\cdots z_n^{y_n}=1, \tag{6}$$

the last identity (6) being by no means as trivial as the former (5) one.

As an example, take a polynomial $f(x_1, x_2)$ of two Boolean variables x_1, x_2. By the theorem 2 we have

$$f(x_1, x_2) = \sum_{(y_1, y_2)} f(y_1, y_2)(1 + x_1 + y_1)(1 + x_2 + y_2) =$$

$$= f(0, 0)(1 + x_1 + x_2 + x_1 x_2) + f(0, 1)(x_2 + x_1 x_2) +$$

$$+ f(1, 0)(x_1 + x_1 x_2) + f(1, 1) x_1 x_2 =$$

$$= f(0, 0) \sum_{\left(z_1^0 z_2^0 = 1\right)} x_1^{z_1} x_2^{z_2} + f(0, 1) \sum_{\left(z_1^0 z_2^1 = 1\right)} x_1^{z_1} x_2^{z_2} +$$

$$+ f(1, 0) \sum_{\left(z_1^0 z_2^0 = 1\right)} x_1^{z_1} x_2^{z_2} + f(1, 1) \sum_{\left(z_1^1 z_2^1 = 1\right)} x_1^{z_1} x_2^{z_2}.$$

By rearranging the terms we get

$$f(x_1, x_2) = x_1^0 x_2^0 \sum_{(0^{y_1} 0^{y_2} = 1)} f(y_1, y_2) + x_1^0 x_2^1 \sum_{(0^{y_1} 1^{y_2} = 1)} f(y_1, y_2) +$$

$$+ x_1^1 x_2^0 \sum_{(1^{y_1} 0^{y_2} = 1)} f(y_1, y_2) + x_1^1 x_2^1 \sum_{(1^{y_1} 1^{y_2} = 1)} f(y_1, y_2) =$$

$$= f(0, 0) + (f(0, 0) + f(0, 1) x_2 + (f(0, 0) + f(1, 0)) x_1 +$$

$$+ (f(0, 0) + f(0, 1) + f(1, 0) + f(1, 1)) x_1 x_2.$$

In the theory of functions in manyvalued logics the same problem arises. Which function should be considered to be the proper generalization of the implication? This question is now easy to answer: The function which has the properties of the exponential function should be denoted as the implication. So we define: $y \rightarrow x$ means x^y where $x^0 = 1$ for each $x \equiv 0, 1, \ldots, k-1$ and $x^y = x \cdot x \ldots x$ (y times) for $z \neq 0$. The problem of the interpolation could be solved in the way we have done it for the case of two valued logic. More detailed description of these problems could be done in a separate article.

LITERATURE

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