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ROTATION-FREE SOLUTIONS WITH POSITIVE INFIMUM OF THE EQUATION $\Delta u = Pu$ IN A NEIGHBORHOOD OF A SINGULARITY OF THE DENSITY P.

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Abstract. It is a classical result of L. Myrberg that there always exists a nonnegative solution of $\Delta u = Pu$, $P \geqslant 0$, on an arbitrary open Riemann surface, and it has been conjectured that there always exists a solution with positive infimum. In this note it is determined that there always exists a rotation-free solution of $\Delta u = \frac{1}{r^\alpha} u$, $\alpha \geqslant 1$, with positive infimum in a neighborhood of the origin.

1. The study of solutions to the elliptic partial differential equation

$$\Delta u = Pu, \tag{1}$$

$P \geqslant 0$, $P \not\equiv 0$, was initiated by Brelot [1] in 1931. In this paper, Brelot investigated the behavior of solutions of (1) in a neighborhood of a singularity of the density P. Of special interest to us will be the positive solutions of (1), and in the aforementioned work we find a characterization of positive bounded solutions for certain classes of rotation-free densities. The subject lay dormant for many years until Ozawa [8] in 1952 rekindled the study of solutions of (1) on arbitrary Riemann surfaces. Subsequently, the topic was pursued by many others, among them Myrberg [4], Royden [10], and Nakai [5, 6, 7]. It was Myrberg who first demonstrated the existence of a Green's function for $\Delta u = Pu$ on a Riemann surface R, and this implies that there always exists a positive solution of (1) on R. Pertinent to our study here is the elegant paper of Nakai [5] dealing with the study of positive minimal solutions of (1). This extends the theory of Martin [3] who introduced the important notion of minimal harmonic functions. A direct consequence of the generalized theory is that the dimension of the space of positive solutions of (1) on an open Riemann surface equals the number of minimal points of the Martin ideal boundary.

More recently the question has arisen whether there always exists solutions of (1) with positive infimum. In this vein, it has been conjectured by Ozawa (cf. Nakai [6]) that for pairs $(R, P) \in O_{PB}$ (the class of Riemann surfaces on which the only bounded solution of $\Delta u = Pu$ is the constant 0), there exists an Evans solution e for the equation (1) (the converse is immediate). Since $e \in PI(R)$, that is has positive infimum on R, one is lead to question whether O_{PI} is void or not. Furthermore, Nakai has shown in [7] that for any open Riemann surface there always exists a smooth density $P \not\equiv 0$ such that (1) has an Evans solution. The null class O_{PI} also has relevance to Φ-bounded solutions of (1) (cf. Schiff [11]), and if O_{PI} is indeed void, then the condition $\overline{\lim}_{t\to\infty} \frac{\Phi(t)}{t} < \infty$ implies the existence of a Φ-bounded solution on an arbitrary open Riemann surface.

In this note we consider rotation-free densities of the form $P(r) = \frac{1}{r^\alpha} (\alpha \geqslant 1)$ in the region $\Omega: 0 < |z| < 1$. It will be shown that for this type of density one can always find a PI-function on Ω. Even for this rather simple region it is still unknown whether there is such a function for an arbitrary density $P \geqslant 0$, $P \not\equiv 0$. A partial answer in the affirmative has been given by Nakai for all rotation-free densities (oral communication).

One might mention in passing the problem of the existence of nonrotation-free positive functions on Ω even for quite simple densities like $P(r) = \frac{1}{r^n}$, n a positive integer. This appears to be a rather difficult matter, and only very partial results have been obtained thus far by the author.

2. In this section we consider solutions of

$$\Delta u = \frac{1}{r^n} u, \tag{2}$$

for $r > 0$, $n = 0, 1, 2, \ldots$. Then we obtain the ordinary differential equation

$$r^2 u'' + ru' - r^{2-n} u = 0. \tag{3}$$

This can be solved by conventional means for $n = 0, 1, 2$, as $r = 0$ is a regular singular point of (3). Since the indicial polynomial has only one root we obtain the following results.

$$u_n(r) = \begin{cases} -(r\sigma(r) + (\log r)\, v(r)), & \text{where } v(r) = \sum_{k=0}^{\infty} \frac{r^k}{2^{2k}(k!)^2} \text{ if } n=0 \text{ is a regular solution of (2), } \sigma(r) \text{ a convergent power series for } r \geqslant 0, \\ -(r\tau(r) + (\log r)\, w(r)), & \text{where } w(r) = \sum_{k=0}^{\infty} \frac{r^k}{(k!)^2} \text{ if } n=1 \text{ is a regular solution of (2), } \tau(r) \text{ a convergent power series for } r \geqslant 0, \\ r^{-1},\ r \text{ a regular solution if } n=2. \end{cases} \tag{4}$$

Note that for $n = 0{,}1$, we have a Picard-type principle in that the singularity at the origin has a logarithmic behavior.

One interesting observation is that for $n = 2$, if we multiply the density $P(r) = \frac{1}{r^2}$ by some constant c^2, $c > 0$, we obtain solutions of (2) of the form $\frac{1}{r^c}$, r^c. This means that a slight alteration of the density can greatly affect the behavior of the resulting solutions.

In the case $n=3, 4, 5, \ldots$ we obtain the unbounded solution

(5) $$u_n(r)=\sum_{k=0}^{\infty}\frac{r^{-k(n-2)}}{(n-2)^{2k}(k!)^2},$$

which converges for all $r>0$. Note that for $n=0, 1$, the previous bounded solutions are obtained.

3. We now seek to find *PI*-functions on Ω for $P(r)=1/r^\alpha$ $\alpha\geqslant 1$. Observe that for these values of α, there exists an integer $n\geqslant 1$ such that

(6) $$\frac{1}{r^n}\leqslant\frac{1}{r^\alpha}\leqslant\frac{1}{r^{n+1}}.$$

Of prime importance will be the following maximum principle (cf. e.g. Royden [10]):

Proposition. Let u be a solution of $\Delta u=Pu$ in a compact region D, and let v be a solution of $\Delta v=Qv$ in D with $Q\leqslant P$. If $0\leqslant u\leqslant v$ on ∂D, then $0\leqslant u\leqslant v$ in D.

Since $0<u_n<u_{n+1}$, $n=1, 2$ in some sufficiently small punctured disk D_n, it shall be necessary to normalize these solutions in the following manner. Denote by w_n the solution to the Dirichlet problem for $\Delta u=\frac{1}{r^n}u$ in D_n, with $w_n|\partial D_n=u_n|\partial D_n$. Here ∂D_n denotes the smooth outer boundary of D_n. This may be done for continuous densities P on D_n which have a singularity at the origin (cf. Brelot [1]). Then w_n is positive and bounded, and the solution u_n-w_n is the desired normalization. It is not difficult to see that $u_{n+1}-u_n$ is unbounded as $r\to 0$, and hence $u_{n+1}-w_{n+1}>u_n-w_n$ in a sufficently small punctured disk, with equality holding on the outer boundary of the disk. In the sequel we shall denote the normalized solutions again by u_n, $n=1, 2$, and without loss of generality we may write $u_n<u_{n+1}$ in Ω with $u_n|\partial\Omega=u_{n+1}|\partial\Omega=0$.

Turning to the remaining cases, $n=3, 4, 5, \ldots$, from (5) one notes that the solutions u_n exhibit the following behavior. On $\partial\Omega$, $u_{n+1}<u_n$, but then for some radius r_n with $0<r_n<1$, $u_n(r_n)=u_{n+1}(r_n)$, and for all positive $r<r_n$, $u_n(r)<u_{n+1}(r)$. As above we take $\{z: 0<|z|<r_n\}$ to be Ω for each $n\geqslant 3$, $u_n|\partial\Omega=u_{n+1}|\partial\Omega=0$.

For $\alpha\geqslant 1$ we consider the solutions u_n, u_{n+1} for which n satisfies (6). Since $du_n/dr<du_{n+1}/dr$ on $\partial\Omega$, we can select a number δ_n such that $du_n/dr_n<\delta_n<du_{n+1}/dr$ on this boundary. Then the mixed Dirichlet problem is solved for $\Delta u=\frac{1}{r^\alpha}u$ in $\Omega_m=\left\{z:\frac{1}{m}<|z|<1\right\}$, where $\{\Omega_m\}_{m=2}^{\infty}$ is an exhaustion of Ω. The solution $u_m{}^\alpha$ satisfies

$$u_m^\alpha\Big|\partial\Omega=0,\quad \frac{du_m^\alpha}{dr}\Big|\partial\Omega=\delta_n,$$

(cf. Ito [2]). In view of (6), the maximum principle yields

$$u_n<u_m^\alpha<u_{n+1}$$

in Ω_m. Furthermore, letting $m \to \infty$,

$$u_m^\alpha \nearrow u^\alpha, \quad u^\alpha | \partial \Omega = 0,$$

and u^α is a solution of $\Delta u = \frac{1}{r^\alpha} u$ in Ω. Finally, $u^\alpha(r) \to \infty$ as $r \to 0$. As a consequence the equation $\Delta u = \frac{1}{r^\alpha} u$, $\alpha \geqslant 1$, has a solution with positive infimum in a neighborhood of the origin.

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