PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 13 (27), 1972, pp. 101—103

OBSERVATION SPACE AND OBSERVABLES IN CLASSICAL MECHANICS

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(Communicated February 17, 1971)

1. Introduction

In an already classical paper G. Birkhoff and J. von Neumann [1] defined the observation space of a mechanical system S relating to the mechanical quantity F as one-dimensional real Euclidean space E_1. Any experimental predicate relating to the mechanical quantity F can be associated either with an element of the set of all subsets ε_1 of the observation space E_1 or with an element of the set of all subsets ε_{2n} of the phase space E_{2n}, where n is the number of degrees of freedom of the system S. Each of two sets $\mathcal{L}_1^F$ and $\mathcal{L}_{2n}$ defined on the observation and the phase space respectively, consisting of elements ono-to-one associated with experimental predicates is called the logic of classical mechanics. In generally accepted interpretation, the logic of classical mechanics is a Boolean σ-algebra [2]. In this case, mechanical quantities (observables) are defined as σ-homomorphisms of the σ-algebra $\mathcal{L}_1^F$ into the σ-algebra $\mathcal{L}_{2n}$. These σ-homomorphisms preserve set-theoretical complement and inclusion (which are the analogues of logical negation and implication) and are isomorphic to the equivalence classes of real measurable functions defined on the phase space E_{2n}[3.] (The equivalence class consists of all functions distinguishable only on a set which belongs to the σ-ideal $I_{2n} \subseteq \varepsilon_{2n}$ of sets of zero Lebesgue measure). The purpose of this paper is to show that:

it is possible to find in the observation and phase space two Boolean complete algebras $\mathcal{L}_1^F$ and $\mathcal{L}_{2n}$ such that the set of all σ-homomorphisms of the $\mathcal{L}_1^F$ into the $\mathcal{L}_{2n}$ is isomorphic to the previously mentioned equivalence classes of real measurable functions defined on the phase space. Thus, we can assume that the logic of classical mechanics is a Boolean complete algebra and still retains the definition of mechanical quantities as σ-homomorphisms which can be, in this case too, identified with generally accepted definition of mechanical quantities as real measurable functions defined on the phase space.

2. Complete algebras $\mathcal{L}_1^P$ and $\mathcal{L}_{2n}$.

Let E_{2n} be a $2n$-dimensional real Euclidean space, ε_{2n} the set of all subsets of the space E_{2n}, $\mathcal{A}_{2n} \subseteq \varepsilon_{2n}$ Lebesgue σ-algebra, L_{2n} (complete) Lebesgue measure defined on the σ-algebra $\mathcal{A}_{2n}$, $I_{2n} \subseteq \mathcal{A}_{2n}$ σ-ideal of sets of zero Lebesgue measure and P_{2n} a probability defined on σ-algebra $\mathcal{A}_{2n}$ equivalent to the

Lebesgue measure L_{2n}. (This means that there is no physical experiment in classical mechanics with ideal precision because of zero probability of any set consisting of only one point of the phase space). The σ-ideal I_{2n} is, obviously, at the same time the σ-ideal of the zero probability sets. Since two elements of the σ-algebra $\mathcal{A}_{2n}$ are mutually equivalent if their symmetric difference belongs to the σ-ideal I_{2n} and the probabilities of equivalent elements are equal, the equation

$$\overline{P}_{2n}[A_{2n}]_{I_{2n}} = P_{2n} A_{2n}$$

defines on the factor-algebra $\mathcal{L}_{2n} = A_{2n}/I_{2n}$, which is a Boolean complete algebra [4], a strictly positive probability $\overline{P}_{2n}\cdot([A_{2n}]_{I_{2n}}$ is the class of equivalence to which belongs the element A_{2n}). In this way we have defined the Boolean complete algebra $\mathcal{L}_{2n}$ with a strictly positive probability $\overline{P}_{2n}$. It is now necessary to define a Boolean complete algebra $\mathcal{L}_1^F = \mathcal{A}_1/I_1$ and to show that the σ-homomorphisms of the complete algebra $\mathcal{L}_1^F$ into the complete algebra $\mathcal{L}_{2n}$ are isomorphic to the previously mentioned classes of real measurable functions defined on the phase space. If such a σ-ideal I_1^F existed, the definition of this σ-ideal could not be based on the Lebesgue measure as we would find the experimental propositions relating to "yes-no" experiments in the class of contradictions. As we are going to show, this σ-ideal can be defined by probabilities which are equivalent to the Lebesgue measure.

Any real valued function F defined on the space E_{2n} defines by the equation

$$F^{-1}(A_1) = \{e_{2n};\quad 1\cdot e_{2n} \in E_{2n},\quad 2\cdot F(e_{2n}) \in A_1\quad 3\cdot A_1 \in \varepsilon_1\}$$

a real (complete) homomorphism of the set of all subsets ε_1 into the set of all subsets ε_{2n}. For example, $I_{A_{2n}}^{-1}\{1\} = A_{2n}$ ($I_{A_{2n}}$ is the indicator function of the set A_{2n}). If the function F is a measurable function, then the homomorphism F^{-1} maps the σ-algebra $\mathcal{A}_1$ into the σ-algebra $\mathcal{A}_{2n}$ By the equation

$$P_1^F A_1 = P_{2n} F^{-1}(A_1)$$

we shall define the probability on the measurable space $(E_1, \mathcal{A}_1)$. If $A_1 = \{1\}$ and $F = I_{A_{2n}}$, where A_{2n} is a non-zero probability set (such sets do exist) then the probability of the set A_1 is different from zero although the set A_1 has zero Lebesgue measure. Therefore, the probability P_1^F is not equivalent to Lebesgue measure L_1. If $I_1^F \subseteq \mathcal{A}_1$ is the σ-ideal of the sets of zero probability with respect to the probability P_1^F, then the homomorphisms F^{-1} maps the σ-ideal I_1^F into the σ-ideal I_{2n}. Since the probabilities of the elements which belong to the same equivalence class are equal, by the equation

$$\overline{P}_1^F [A_1]_{I_1^F} = P_1^F A$$

we shall define on the factor-algebra $\mathcal{L}_1 = \mathcal{A}_1/I_1^F$ a strictly positive probability $\overline{P}_1^F$. Therefore, every real measurable function F defines a Boolean complete algebra $\mathcal{L}_1^F$ with a strictly positive probability $\overline{P}_1^F$. It is evident that $\overline{P}_1^F[A_1]_{I_1^F} = \overline{P}_{2n}[F^{-1}(A_1)]_{I_{2n}}$ and since the probability P_{2n} is equivalent to Lebesgue measure L_{2n}, there exists such a class of measurable functions $[G]_{I_{2n}}$ [5], so that for any function G from the class

$$P_{2n} A_{2n} = \int_{\mathcal{A}_{2n}} G dL_{2n}.$$

In Statistical Mechanics we have $G\sim|\operatorname{grad} H|^{-1}$ or $G\sim e^{-\frac{H}{kT}}$. Since the homomorphism F^{-1} maps the σ-ideal I_1^F into the σ-ideal I_{2n}, by the equation

$$H_F[A_1]_{I_1^F}=[F^{-1}(A_1)]_{I_{2n}}$$

we shall define a real σ-homomorphism of the Boolean complete algebra $\mathcal{L}_1^F$ into the Boolean complete algebra $\mathcal{L}_{2n}$. As the Euclidean space is a complete metric space, the inverse theorem is valid too [6]. Namely, each real σ-homomorpism defines a class of equivalent real measurable functions. By these σ-homomorphisms we shall define mechanical quantities. If we define the operations with σ-homomorphisms in the usual manner [7], then the concept of the real σ-homomorphism has the same mathematical content as the concept of the real measurable function and these two mathematical objects can be identified.

3. Concluding remarks

In the present paper we postulate that the logic of the classical mechanics is a Boolean complete algebra. We postulate that there exists one-to one correspondence between the experimental predicates concerning a mechanical quantity F (and mechanical system S) and the elements of the complete algebra $\mathcal{L}_1^F$ as well as the elements of the complete algebra $\mathcal{L}_{2n}$. We define the mechanical quantities (observables) as σ-homomorphisms of the complete algebra $\mathcal{L}_1^F$ into the complete algebra $\mathcal{L}_{2n}$. These σ-homomorphisms are isomorphic to real measurable functions defined on the phase space of the mechanical system. We restrict ourselves to the probabilities equivalent to the Lebesgue measure in order to agree with idea that there is no physical experiment with ideal precision.

4. Abstract

It is shown that in the observation as well as in the phase space of a mechanical system two Boolean complete algebras can be found such that the set of all σ-homomorphisms of the one algebra into the other one is isomorphic to the set of all real measurable functions defined on the phase space with values in the observation space. The mechanical quantities are defined by these σ-homomorphisms and two Boolean complete algebras are interpreted as the logic of the classical mechanics.

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