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A VARIATIONAL APPROACH TO THE THEORY OF TEMPERATURE BOUNDARY LAYER

M. Pavlović

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1. *Introduction*

In investigations of dynamical and temperature boundary layer there are several well developped "exact" and approximative methods. But nothing is done concerning a variational approach to this group of problems, mainly it is not possible to derive the corresponding equations of the process in consideration. Especially it is not possible to find exact Lagrangian density, by the help of which the governing differential equations of the process could be derived. However the usefulness of variational principles of Hamilton's type for problems in mechanics is well established. To eliminate this difficulty of variational description of irreversible processes it was proposed a certain number of new variational formulations. Generally speaking, we can briefly classify these new variational formulations in the following groups:

a) Formulations in which the basic principles of variational calculus are violated (Glansdorff, Prigogine's method of Local Potential).

b) Bateman's variational principle with conjugated functions of field, without any physical interpretation.

c) Variational approach which was introduced in the references [1], [2], [3] and [4], where, resulting of variation of a Lagrangian, more composite differential equations were obtained than real differential equations of the process. Lagrange's density in such a formulation includes also one parameter tending to zero after the process of variation is finished. So, after the limiting process is done, the exact differential equations of the process are obtained. This variational principle was applied to the large number of irreversible processes, as: nonlinear heat transfer, radiation problems, nonstationary dinamical boundary layer, boundary layer of non — Newtonian viscous power law fluids etc.

The aim of this study is to demonstrate that it is possible to enlarge this variational formulation over the theory of convection heat transfer problems.

In the variational formulation of the problem, the chain-systems conception is used, introduced in mechanics of discreet systems by I. S. Aržanyh, in the reference [5]. Application of this principle to the theory of temperature boundary layer shows:

— The method is easy and quickly gives results

— The variational principle is especially favourable for application of standard Kantorovich's direct method of variational calculus.

— The results obtained in such a way are in an excellent agreement with results obtained by means of other methods.

2. *Variational principle*

We are going to consider a temperature boundary layer on the flat plate of constant temperature, where the heat transfer is of convectional type over an incompressible fluid of constant viscosity, which is in stationary laminar flow along the plate.

The problem is reduced to solving differential equation of stationary plane flow:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2} \tag{1}$$

with equation of continuity:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{2}$$

and equation of heat balance:

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{\nu}{\sigma}\frac{\partial^2 T}{\partial y^2} \tag{3}$$

So, it is necessary to define three unknown functions u, v and T which have to satisfy following boundary conditions:

$$\text{If } y=0 \quad \text{then } u=v=0 \qquad T=T_w$$
$$\text{If } y=\infty \quad \text{then } U=U_\infty \qquad T=T_\infty.$$

Following the theory of chain-systems*, components u and v are to be separated in the first group and temperature T in the second group of "coordinates" with partial Lagrangians,

$$L^{(1)}=\left\{m\left[\frac{1}{2}u\left(\frac{\partial u}{\partial x}\right)^2+v\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right]-\frac{\nu}{2}\left(\frac{\partial u}{\partial y}\right)^2\right\}e^{x/m} \tag{4}$$

$$L^{(2)}=\left\{m\left[\frac{1}{2}u\left(\frac{\partial T}{\partial x}\right)^2+v\frac{\partial T}{\partial x}\frac{\partial T}{\partial y}\right]-\frac{\nu}{2\sigma}\left(\frac{\partial T}{\partial y}\right)^2\right\}e^{x/m} \tag{5}$$

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* According to I. S. Aržanyh as chain-system is considered such a mechanical system in which the generalized coordinates are separated in k groups:

$$q_{i1},\ q_{i2},\ q_{in_i}, \qquad (i=1,2,\ldots,k)$$

with k partial Lagrangian $L^{(i)}$, $(i=1,2,\ldots,k)$ which, in general case, are depending on time, of all generalized coordinates and of all generalized velocities, so that equations:

$$\frac{d}{dt}\frac{\partial L^{(i)}}{\partial \dot{q}_{i\alpha}}-\frac{\partial L^{(i)}}{\partial q_{i\alpha}}=0 \qquad \begin{matrix}(i=1,2,\ldots,k)\\(\alpha=1,2,\ldots,n_i)\end{matrix}$$

are giving differential equations of considered system.

corresponding to two action integrals.

(6) $$I_1=\iint L^{(1)}\,dx\,dy$$

(7) $$I_2=\iint L^{(2)}\,dx\,dy.$$

Differential equations (1) and (3) can be derived from variational principle:

(8) $$\delta I_1=\delta\iint L^{(1)}\,dx\,dy=0$$

(9) $$\delta I_2=\delta\iint L^{(2)}\,dx\,dy=0$$

where, in the first action, the integral operation of variation relates only to the first group of „coordinates" (u, v) and in the second one to the second group of „coordinates" (T).

If the following natural boundary conditions on $x=l$ are satisfied for arbitrary values of variations of velocity u and temperature T: (see ref. |6|):

(10) $$\left.\frac{\partial L^{(1)}}{\partial\left(\frac{\partial u}{\partial x}\right)}\,\delta u\right|_{x=l}=0$$

(11) $$\left.\frac{\partial L^{(2)}}{\partial\left(\frac{\partial T}{\partial x}\right)}\,\delta T\right|_{x=l}=0$$

then, the variational problem is equivalent with the following system of Lagrange's equations:

(12) $$\frac{\partial L^{(1)}}{\partial u}-\frac{\partial}{\partial x}\,\frac{\partial L^{(1)}}{\partial\left(\frac{\partial u}{\partial x}\right)}-\frac{\partial}{\partial y}\,\frac{\partial L^{(1)}}{\partial\left(\frac{\partial u}{\partial y}\right)}=0$$

(13) $$\frac{\partial L^{(1)}}{\partial v}-\frac{\partial}{\partial x}\,\frac{\partial L^{(1)}}{\partial\left(\frac{\partial v}{\partial x}\right)}-\frac{\partial}{\partial y}\,\frac{\partial L^{(1)}}{\partial\left(\frac{\partial v}{\partial y}\right)}=0$$

(14) $$\frac{\partial L^{(2)}}{\partial T}-\frac{\partial}{\partial x}\,\frac{\partial L^{(2)}}{\partial\left(\frac{\partial T}{\partial x}\right)}-\frac{\partial}{\partial y}\,\frac{\partial L^{(2)}}{\partial\left(\frac{\partial T}{\partial y}\right)}=0.$$

Substituting (4) into (10), (12), (13) and (5) into (11) and (14), and dividing with $e^{x/m}$, we get

(a) $$\left.m\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)\delta u\right|_{x=l}=0$$

(b) $$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+m\left[\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial y}\left(v\frac{\partial u}{\partial x}\right)-\frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2\right]=\nu\frac{\partial^2 u}{\partial y^2}$$

(c) $$m\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right)=0$$

(d) $$m\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)\delta T\Big|_{x=l}=0$$

(e) $$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+m\left[\frac{\partial}{\partial x}\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial y}\left(v\frac{\partial T}{\partial x}\right)\right]=\frac{\nu}{\sigma}\frac{\partial^2 T}{\partial y^2}.$$

After the limiting process is done, i.e $m\to 0$, equations (a) and (d) are identically satisfied for arbitrary values of variations δu and δT; equation (c) also identically satisfied, and equations (b) and (e) are giving differential equations of problems (1) and (3).

Following ref. [3] the continuity equation plays the roll of nonholonomic constraint. (For more details see ref. [3]).

3. *Applications to the temperature boundary layer*

In this section we will apply the variational principle formulated in previous section for obtaining approximative solutions of temperature boundary layer. Let us consider the flat plate of constant temperature T_w. Along the plate flows incompressible fluid. We shall suppose that fluid has constant viscosity. We will suppose also, that motion of the fluid has steady and laminar character. The direct method of variational calculus in the form of partial integration (method of Kantorovich) is the basic tool for obtaining approximative solutions.

Let us take action integrals

(15) $$I_1=\int_0^l\int_0^f L^{(1)}\,dx\,dy$$

(16) $$I_2=\int_0^l\int_0^\delta L^{(2)}\,dx\,dy.$$

In order to get the approximative solution, we are going to take the profile of the component u in the form:

(17) $$u=U_\infty\,\Phi(\eta),$$

where Φ represents the known function of η and

(18) $$\eta=\frac{y}{f(x)}.$$

$f(x)$ is the thickness of dynamical boundary layer. The function $\Phi(\eta)$ has to satisfy the boundary conditions:

(19) $$\begin{array}{llll}\Phi(\eta)=0 & \dfrac{\partial^2\Phi}{\partial\eta^2}=0 & & \eta=0\\[2ex] \Phi(\eta)=1 & \dfrac{\partial\Phi}{\partial\eta}=0 & \dfrac{\partial^2\Phi}{\partial\eta^2}=0 & \eta=1.\end{array}$$

Let us suppose that the velocity component in y – direction has the form

$$v = g(x) N(\eta). \tag{20}$$

Function $N(\eta)$ has to perform the boundary condition

$$N(0) = 0. \tag{21}$$

A deeper analyses shows that the nondimensional ratio of temperatures

$$\theta = \frac{T - T_w}{T_\infty - T_w}$$

and the ratio $\frac{u}{U_\infty}$ can be expressed by means of function of the same form and different arguments. Therefore, we are taking

$$\frac{T - T_w}{T_\infty - T_w} = \Phi(\eta_T),$$

where Φ is the known function of η_T of the same form as in the profile of velocity u (17), and

$$\eta_T = \frac{y}{\delta(x)}. \tag{22}$$

$\delta(x)$ is the thickness of temperature layer. Accordingly, the temperature profile has the form:

$$T = C\Phi(\eta_T) + T_w; \qquad C = T_\infty - T_w. \tag{23}$$

Function $\Phi(\eta_T)$ has to accomplish following boundary conditions

$$\begin{array}{lll} \Phi(\eta_T) = 0 & T = T_w & \eta_T = 0 \\ \Phi(\eta_T) = 1 & T = T_\infty & \eta_T = 1. \end{array} \tag{24}$$

In following considerations we will take the thickness of dynamical boundary layer greater than the temperature boundary layer thickness, namely

$$\frac{\delta(x)}{f(x)} < 1$$

Substituting the profiles (17), (20) and (23) in the action integrals (15) and (16) and integrating with respect to y, we are getting reduced action integrals:

$$I_1 = \int_0^l L_1(f, f_x, g, x, m)\, dx \tag{25}$$

$$I_2 = C^2 \int_0^l L_2(f, f_x, \delta, \delta_x, g, x, m)\, dx \tag{26}$$

with partial Lagrangians

$$L_1 \equiv \left[m \left(\frac{1}{2} \frac{U_\infty^3 f_x^2}{f} A_1 - \frac{g U_\infty^2 f_x}{f} A_2 - \nu \frac{U_\infty^2}{f} A_3 \right] e^{x/m} \tag{27}$$

$$L_2 \equiv \left\{ m \left[U_\infty \delta_x^2 F_1(f, \delta) + g \delta_x F_2(f, \delta) \right] - \frac{\nu}{2\sigma} F_3(\delta) \right\} e^{x/m} \tag{28}$$

where

$$A_1=\int_0^1 \Phi(\eta)\,\Phi'^2(\eta)\,\eta^2\,d\eta; \qquad A_2=\int_0^1 N(\eta)\,\Phi'^2(\eta)\,\eta\,d\eta$$

(29)

$$A_3=\frac{1}{2}\int_0^1 \Phi'^2(\eta)\,d\eta$$

$$F_1(f,\delta)=\frac{1}{\delta}\int_0^1 \Phi\left(\frac{\delta}{f}\,\eta_T\right)\Phi'^2(\eta_T)\,\eta_T^2\,d\eta_T;$$

(30)
$$F_2(f,\delta)=-\frac{1}{\delta}\int_0^1 N\left(\frac{\delta}{f}\,\eta_T\right)\Phi'^2(\eta_T)\,\eta_T\,d\eta_T;$$

$$F_3(\delta)=-\frac{1}{\delta}\int_0^1 \Phi'^2(\eta_T)\,d\eta_T.$$

Functions $f(x)$, $g(x)$ and $\delta(x)$ represent generalized coordinates. Taking care of before mentioned theory of chain systems and connection between f, g and δ and also between u, v and T, we can consider the "generalized coordinates" f and g in the first and δ in the second group; L_1 as partial Lagrangian referring to the first and L_2 partial Lagrangian relating to the second group. Accordingly, varying the reduced action integral (25) we are varying functions f and g and their derivations, and by varying the reduced action integral (26) we are varying only δ and its derivatives.

The conditions of stationarity of reduced action integrals

$$\delta\int_0^l L_1\,dx=0$$

$$\delta\int_0^l L_2\,dx=0$$

correspond to Euler-Lagrange's equations

$$\frac{\partial L_1}{\partial f}-\frac{d}{dx}\,\frac{\partial L_1}{\partial f_x}=0$$

$$\frac{\partial L_1}{\partial g}-\frac{d}{dx}\,\frac{\partial L_1}{\partial g_x}=0$$

$$\frac{\partial L_2}{\partial \delta}-\frac{d}{dx}\,\frac{\partial L_2}{\partial \delta_x}=0.$$

Let us substitute L_1 and L_2 in previous equations, devide with $e^{x/m}$ and accomplish the limiting process when m tends to zero, then we shall obtain from the first and the third equation

(31) $$\frac{U_\infty^3 f_x}{f} A_1 - \frac{g U_\infty^2}{f} A_2 - \nu \frac{U_\infty^2}{f^2} A_3 = 0$$

(32) $$U_\infty \delta_x F_1(f, \delta) + g F_2(f, \delta) - \frac{\nu}{2\sigma} \frac{dF_3}{d\delta} = 0,$$

while the second equation is identically satisfied. Substituting the profiles (17) and (20) in the continuity equation (2) we are getting:

(33) $$g(x) = U_\infty f_x$$

$$\frac{dN(\eta)}{d\eta} = \eta \Phi'(\eta),$$

(34) $$N(\eta) = \int \eta \Phi'(\eta) d\eta + C_1.$$

By substituting (33) into (31) and (32) we get

$$\frac{f_x}{f}(A_1 - A_2) = \frac{\nu}{f^2 U_\infty} A_3$$

$$F_1(f, \delta) \frac{d\delta}{dx} + F_2(f, \delta) \frac{df}{dx} = \frac{d\delta}{2\sigma U_\infty} \frac{dF_3(\delta)}{d\delta}$$

or, multiplying the last equation by δ.

(35) $$f \frac{df}{dx} = \frac{A_3}{A_1 - A_2} \frac{\nu}{U_\infty}$$

(36) $$\delta \cdot F_1(f, \delta) d\delta + \delta \cdot F_2(f, \delta) df = \frac{\nu\delta}{2\sigma U_\infty} \frac{dF_3}{d\delta} dx.$$

By integrating the equation (35) we get the thickness of dynamical boundary layer

(37) $$f = \sqrt{\frac{2A_3}{A_1 - A_2}} \sqrt{\frac{\nu}{U_\infty} x}.$$

Let us suppose that there exists the function $\psi(f, \delta)$ which satisfies following conditions:

(38) $$\frac{\partial \psi}{\partial \delta} = \delta \cdot F_1(f, \delta); \qquad \frac{\partial \psi}{\partial f} = \delta \cdot F_2(f, \delta).$$

Then the equation (36) can be written in the form

(39) $$d\psi(f, \delta) = \frac{\nu\delta}{2\sigma U_\infty} \frac{dF_3}{d\delta} dx.$$

We are going to apply the obtained results to the two cases of velocity and temperature profiles.

Let us consider, in the first place, the case for which is:

(40)
$$\Phi(\eta) = 2\eta - 2\eta^3 + \eta^4$$
$$\Phi(\eta_T) = 2\eta_T - 2\eta_T^3 + \eta_T^4.$$

In this case, according to (34), (21), (29):

(41)
$$N(\eta) = \frac{1}{10}(10\eta^2 - 15\eta^4 + 8\eta^5)$$

$$A_1 - A_2 = 0{,}0472; \qquad 2A_3 = 1{,}484.$$

So the thickness of dynamical boundary layer according to (37) is:

(42)
$$f = 5{,}61\sqrt{\frac{\nu}{U_\infty}x}.$$

From here we get

(43)
$$f^2 = 31{,}48\frac{\nu}{U_\infty}x; \qquad f\frac{df}{dx} = 15{,}74\frac{\nu}{U_\infty}.$$

Substituting (40) and (41) into (30) we get:

$$F_1(f,\delta) = 4\left(\frac{11}{420}\frac{1}{f} - \frac{7}{990}\frac{\delta^2}{f^3} + \frac{31}{15015}\frac{\delta^3}{f^4}\right)$$

$$F_2(f,\delta) = 4\left(-\frac{11}{840}\frac{\delta}{f^2} + \frac{7}{1320}\frac{\delta^3}{f^4} - \frac{124}{75075}\frac{\delta^4}{f^5}\right)$$

$$F_3(\delta) = -\frac{52}{35}\frac{1}{\delta}.$$

Because the function:

$$\psi(f,\delta) = 4\left(\frac{11}{840}\frac{\delta^2}{f} - \frac{7}{3960}\frac{\delta^4}{f^3} + \frac{31}{75075}\frac{\delta^5}{f^4}\right)$$

satisfies conditions (38), equation (39) can be written as

$$4\cdot d\left(\frac{11}{840}\frac{\delta^2}{f} - \frac{7}{3960}\frac{\delta^4}{f^3} + \frac{31}{75075}\frac{\delta^5}{f^4}\right) = \frac{\nu\delta}{2U_\infty\sigma}\frac{52}{35}\frac{1}{\delta^2}dx$$

or, introducing the variable $\xi = \frac{\delta}{f}$:

(44)
$$d\left[f\left(\frac{11}{840}\xi^2 - \frac{7}{3960}\xi^4 + \frac{31}{75075}\xi^5\right)\right] = \frac{13}{70}\frac{\nu}{U_\infty\sigma}\frac{1}{\xi f}dx.$$

Because $\delta < f$ in the preliminary equation, we can ignore the terms

$$\frac{7}{3960}\xi^4 \quad \text{and} \quad \frac{31}{75075}\xi^5$$

and the equation (44) takes the form

$$\frac{11}{840}\xi f \frac{d}{dx}(f\xi^2)=\frac{13}{70}\frac{\nu}{U_\infty \sigma}$$

or

$$\xi^2 f\frac{df}{dx}+2\xi^2 f^2\frac{d\xi}{dx}=\frac{156}{11}\frac{\nu}{U_\infty\sigma}.$$

Substituting f^2 and $\frac{df}{dx}$ from (43) we get

$$\xi^3+4\xi^2 x\frac{d\xi}{dx}=\frac{1}{k\sigma}; \qquad k=\frac{11\cdot 15{,}74}{156}.$$

By introducing a new variable $z=\xi^3$, the preceding equation becomes

$$z+\frac{4}{3}x\frac{dz}{dx}=\frac{1}{k\sigma}.$$

Integrating, we obtain

$$x^{3/4}\left(z-\frac{1}{k\sigma}\right)=C_1.$$

When $x=0$, $\delta=0$, $\xi=0$ and $z=0$, we have $C_1=0$ and

$$z=\frac{1}{k\sigma}=\xi^3=\frac{\delta^3}{f^3}.$$

From here we obtain the thickness of temperature boundary layer:

$$\delta=\frac{f}{\sqrt[3]{k\sigma}}=\frac{5{,}61\sqrt{\frac{\nu}{U_\infty}x}}{\sqrt[3]{k}\sqrt[3]{\sigma}}. \tag{45}$$

Connection between the local coefficient of heat transfer α_x and coefficient of conductivity λ is given by means of relation

$$\alpha_x=\frac{\lambda}{T_\infty-T_w}\left(\frac{\partial T}{\partial y}\right)_{y=0}.$$

In this case we have

$$\left(\frac{\partial T}{\partial y}\right)_{y=0}=C\left[\frac{\Phi'(\eta_T)}{\delta}\right]_{\eta_T=0}=2C\frac{1}{\delta}=2(T_\infty-T_w)\cdot\frac{1}{\delta}$$

so, we get:

$$\alpha_x=\frac{2\lambda}{\delta}=\lambda\frac{2\sqrt[3]{k}}{5{,}61}\sqrt[3]{\sigma}\sqrt{\frac{U_\infty}{\nu x}}.$$

The local Nusselt's number is

$$Nu_x=\frac{\alpha_x x}{\lambda}=\frac{2\sqrt[3]{k}}{5{,}61}\sqrt[3]{\sigma}\sqrt[3]{Re_x}$$

$$Nu_x=0{,}369\sqrt[3]{\sigma}\sqrt{Re_x}. \tag{46}$$

In the second example we shall take the profiles of velocity and temperature in the form:

(47)
$$\Phi(\eta)=\frac{3}{2}\eta-\frac{1}{2}\eta^3$$
$$\Phi(\eta_T)=\frac{3}{2}\eta_T-\frac{1}{2}\eta_T^3.$$

Following (34), (21) and (29) we have

$$N(\eta)=\frac{3}{4}\eta^2-\frac{3}{8}\eta^4$$
$$A_1-A_2=\frac{21}{320}; \qquad 2A_3=\frac{6}{5}.$$

So, according to (37), the thickness of dynamical boundary layer is:

(49)
$$f=\sqrt{\frac{128}{7}}\sqrt{\frac{\nu}{U_\infty}x}.$$

From here, we have

(50)
$$f\frac{df}{dx}=\frac{64}{7}\frac{\nu}{U_\infty}.$$

Substituting (47) and (48) into (30) we get

$$F_1(f,\delta)=\frac{9}{32}\left(\frac{1}{2}\frac{1}{f}-\frac{1}{15}\frac{\delta^2}{f^3}\right)$$
$$F_2(f,\delta)=\frac{9}{32}\left(-\frac{1}{4}\frac{\delta}{f^2}+\frac{1}{20}\frac{\delta^3}{f^4}\right)$$
$$F_3(\delta)=-\frac{6}{5}\frac{1}{\delta}.$$

Because the function

$$\psi(f,\delta)=\frac{9}{32}\left(\frac{1}{4}\frac{\delta^2}{f}-\frac{1}{60}\frac{\delta^4}{f^3}\right)$$

satisfies all conditions (38), the equation (39) we may write

$$\frac{9}{32}d\left(\frac{1}{4}\frac{\delta^2}{f}-\frac{1}{60}\frac{\delta^4}{f^3}\right)=\frac{\nu\delta}{2U_\infty\sigma}\frac{6}{5}\frac{1}{\delta^2}dx.$$

Introducing a new variable $\xi=\frac{\delta}{f}$, the preceding equation takes following form

$$d\left[f\left(\frac{1}{4}\xi^2-\frac{1}{60}\xi^4\right)\right]=\frac{32}{15}\frac{\nu}{U_\infty\sigma}\frac{1}{\xi f}dx.$$

Because $\delta < f$, the member $\frac{1}{60}\xi^4$ may be ignored, so we get:

$$\frac{1}{4} f \xi \frac{d}{dx}(f \xi^2) = \frac{32}{15} \frac{\nu}{U_\infty \sigma}$$

or

$$\xi^3 f \frac{df}{dx} + 2 \xi^2 f^2 \frac{d\xi}{dx} = \frac{128}{15} \frac{\nu}{U_\infty \sigma}.$$

Using f and $\frac{df}{dx}$ from (49) and (50) we obtain:

$$\xi^3 + 4 \xi^2 x \frac{d\xi}{dx} = \frac{1}{k_1 \sigma}; \qquad k_1 = \frac{15}{14}.$$

Applying the same method as in the preceding case, we get the thickness of temperature boundary layer:

$$\delta = \frac{f}{\sqrt[3]{k_1 \sigma}} = \frac{\sqrt{\frac{128}{7}} \sqrt{\frac{\nu}{U_\infty} x}}{\sqrt[3]{k_1} \sqrt[3]{\sigma}}.$$

For this temperature profile we have

$$\left(\frac{\partial T}{\partial y}\right)_{y=0} = \left[C \frac{\Phi'(\eta_T)}{\delta}\right]_{\eta_T=0} = \frac{3}{2} C \frac{1}{\delta} = \frac{3}{2}(T_\infty - T_w)\frac{1}{\delta}$$

$$\alpha_x = \frac{3}{2} \frac{\lambda}{\delta} = \lambda \frac{3 \sqrt[3]{k_1} \sqrt{7}}{2 \sqrt{128}} \sqrt[3]{\sigma} \sqrt{\frac{U_\infty}{\nu x}}.$$

Now, the local Nusselt's number is:

$$Nu_x = 0,358 \sqrt[3]{\sigma} \sqrt{Re_x}.$$

The value of local Nusselt's number obtained by means of exact method [6] is:

$$Nu_x = 0,332 \sqrt[3]{\sigma} \sqrt{Re_x}.$$

Nomenclature

T — Temperature

T_w — Temperature of plate

T_∞ — Temperature of fluid flow for $x=0$

u — Longitudinal component of velocity

v — Transversal component of velocity

U_∞ — Velocity of fluid flow for $x=0$

f — Thickness of dynamical boundary layer

α — Coefficient of heat transfer

δ — Thickness of temperature boundary layer

λ — Coefficient of conductivity of heat

ν — Kinematical viscosity

N — Nusselt's number

Re — Reynold's number

σ — Prandtl's number

REFERENCES

[1] B. Vujanović: *An Approach to Linear and Nonlinear Heat-Transfer, Problem Using a Lagrangian*, AIAA JOURNAL Vol. 9, January 1971, pp 131—134.

[2] B. Vujanović, A. M. Strauss, *Heat Transfer with Nonlinear Boundary Conditions via a Variational Principle*, AIAA JOURNAL, Vol. 9, № 2, February 1971, pp 327—330.

[3] M. Pavlović, Dj. Djukić, B. Vujanović: *A Variational Approach to the Theory of Boundary Layer*, Information of Mathematical institute — in print.

[4] Dj. Djukić: *Stationary Planar Laminary Boundary Layer of Non-Newton's Liquid with Gradual Rheological Equation of State*, (Dissertation 1971).

[5] И. С. Аржаных, „*О цепных системах теории нелинейных колебаний*", Издательство Академии Наук УССР — Киев 1963. стр. 59—72.

[6] R. S. Schechter: *The Variational Method in Engineering*, MC GRAW-HILL Book Company, New York, 1967, p. 28

[7] А. Жукаускас, И. Жюгжда, *Теплотдача в ламинарном потоке жидкости*. Издательство „Минтис" Вильнюс — 1969.