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EXPONENTIALLY COMPLETE SPACES II

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1. Introduction. Let $\mathcal{K}=\{K, M\}$ be the category whose objects K are all compact Hausdorff spaces and morphisms M all continuous mappings of these spaces. For $X \in K$, let $\exp(X)$ denote the set of all non-empty closed subsets of X taken with the finite topology and for $f: X \to Y$ in M, let $\exp(f): \exp(X) \to \exp(Y)$ be the mapping defined by $\exp(f)(F)=f(F)$. Then $\exp: \mathcal{K} \to \mathcal{K}$ becomes a covariant functor.

Let $\{X, \pi, A\}$ be an inverse system over the directed set A. Since exp is covariant, $\{\exp(X), \exp(\pi), A\}$ will also be an inverse system.

The fact that two spaces $\varprojlim \{\exp(X), \exp(\pi)\}$ and $\exp(\varprojlim \{X, \pi\})$ are homeomorphic was proved by S. Sirota [3]. Let $\{Y, \rho, B\}$ be another inverse system in $\mathcal{K}$ over the directed set B. If $\Phi: \{X, \pi\} \to \{Y, \rho\}$ is a mapping of these two systems, then it obviously defines a mapping $\exp(\Phi): \{\exp(X), \exp(\pi)\} \to \{\exp(Y), \exp(\rho)\}$. So we have two induced mappings $\exp(\varprojlim \Phi)$ and $\varprojlim \exp(\Phi)$ and we will prove here that they are the same up to the composition with homeomorphisms (see 2.3). This completes the above mentioned result of S. Sirota.

For $X \in K$, denote $\exp(X)$ by $X^{(1)}$, and for $n=2, 3, \ldots$, let

$$X^{(n)}=\exp(X^{(n-1)}).$$

For $F^{(1)} \in X^{(2)}$, let

$$u(F^{(1)})=\bigcup\{x: x \in F \in F^{(1)}\},$$

so we have a continuous mapping $u: X^{(2)} \to X^{(1)}$. Let

$$u^{(n)}=\exp(u^{(n-1)}): X^{(n+1)} \to X^{(n)}, \qquad n=2, 3, \ldots,$$

where $u^{(1)}=u$. For $f: X \to Y$, let

$$f^{(n)}=\exp(f^{(n-1)}), \qquad n=2, 3, \ldots, \; f^{(1)}=\exp(f).$$

So we get

$$\begin{array}{cccccc}
\cdots \leftarrow & X^{(n)} & \xleftarrow{u^{(n)}} & X^{(n+1)} & \leftarrow \cdots & X^{(\omega)}=\varprojlim \{X^{(n)}, u^{(n)}\} \\
& \downarrow f^{(n)} & & \downarrow f^{(n+1)} & & \downarrow f^{(\omega)}=\varprojlim (f^{(n)}) \\
\cdots \leftarrow & Y^{(n)} & \xleftarrow[u^{(n)}]{} & Y^{(n+1)} & \leftarrow \cdots & Y^{(\omega)}=\varprojlim \{Y^{(n)}, u^{(n)}\}
\end{array}$$

A topological space is called *exponentially complete* if $x \approx \exp(X)$. In [2], we have shown that $X^{(w)}$ is exponentially complete. Here we will prove that $f^{(\omega)}$ is also *exponentially complete* in the sense that $f^{(\omega)}$ is equal to $\exp(f^{(\omega)})$ up to the composition with homeomorphisms.

2. If E is any subset of X, then we write

$$\rangle E \langle = \{x \in X\{X_\alpha : \alpha \in A\} : x(\alpha) \in E\}.$$

Now we will state a result in a form which is convenient to us and we will give a proof. We suppose that all spaces and all mappings that we consider in this paper belong to $\mathcal{K}$.

2.1. *Let* $X_\infty = \varprojlim \{X, \pi\}$. *Then, a non-empty closed set* $F \subseteq X_\infty$ *can be written as* $F = \cap\{\rangle \pi_\alpha(F) \langle\} \cap X_\infty$, *where all* $\pi_\alpha(F)$ *are closed and for* $\alpha > \alpha'$, $\pi_{\alpha\alpha'}(\pi_\alpha(F)) = \pi_{\alpha'}(F)$. *If* $F = \cap\{\rangle T_\alpha \langle\} \cap X_\infty$, *where all* T_α *are closed and such that* $\pi_{\alpha\alpha'}(T_\alpha) = T_{\alpha'}$, *then* $\pi_\alpha(F) = T_\alpha$.

Proof. It is easily seen that $\pi_\alpha(F)$ are closed, $\pi_{\alpha\alpha'}(\pi_\alpha(F)) = \pi_{\alpha'}(F)$ and $F \subseteq \cap\{\rangle \pi_\alpha(F) \langle\} \cap X_\infty$. So suppose $x \in \cap\{\rangle \pi_\alpha(F) \langle\} \cap X_\infty$. and let $\rangle U_{\alpha_1} \langle \cap \cdots \cap \rangle U_{\alpha_n} \langle$ be a typical basic neighborhood of x. Take $\bar{\alpha} > \alpha_i$, $i = 1, \ldots, n$. Choose $y \in F$ such that $y(\bar{\alpha}) = x(\bar{\alpha})$. Then $y(\alpha_i) = x(\alpha_i)$ and $y \in \rangle U_{\alpha_1} \langle \cap \cdots \cap \rangle U_{\alpha_n} \langle$. Therefore, $x \in \bar{F} = F$.

To prove the second part, let us note that $T_\infty = \varprojlim \{T_\alpha : \pi \mid T_\alpha\} \subseteq X_\infty$ and $\pi_\alpha(T_\infty) = T_\alpha$ ([1]). Obviously

$$\pi_\alpha(F) \supseteq \pi_\alpha\big((\cap\{\rangle T_\alpha \langle\}) \cap T_\infty\big) = \pi_\alpha(T_\infty) = T_\alpha$$

and this together with $\pi_\alpha(F) \subseteq T_\alpha$ concludes the proof.

Now let Π_α be the natural projection of $X\{\exp(X_\alpha) : \alpha \in A\}$ onto $\exp(X_\alpha)$. Consider the mapping

$$H : \exp(\varprojlim \{X, \pi\}) \to \varprojlim \{\exp(X), \exp(\pi)\}$$

defined by

$$\Pi_\alpha \circ H(F) = \pi_a(F) = \exp(\pi_\alpha)(F).$$

H is obviously continuous and by the fist part of 2.1 H is $1-1$ and by the second onto. So H is a homeomorphism and we have

2.2. $H : \exp(\varprojlim X, \pi\}) \approx \varprojlim \{\exp(X), \exp(\pi)\}$ ([3]).

Now we can prove.

2.3. *Given a mapping of inverse systems.*

$$\Phi : \{X, \pi, A\} \to \{Y, \rho, B\}$$

Then, there exist two homeomorphisms H *and* K *such that the diagram*

$$\begin{array}{ccc}
\exp(\varprojlim \{X, \pi\}) & \xrightarrow{H} & \varprojlim \{\exp(X), \exp(\pi)\} \\
\Big\downarrow {\scriptstyle \exp(\varprojlim \Phi)} & & \Big\downarrow {\scriptstyle \varprojlim \exp(\Phi)} \\
\exp(\varprojlim \{Y, \rho\}) & \xrightarrow[K]{} & \varprojlim \{\exp(Y), \exp(\rho)\}
\end{array}$$

commutes.

P r o o f. Let H and K be homeomorphisms from 2.2, related to $\{X, \pi\}$ and $\{Y, \rho\}$ respectively and $\Pi_\alpha \circ H(F) = \pi_\alpha(F)$, $\alpha \in A$, for $F \in \exp(\varprojlim\{X, \pi\})$. So, we have, according to the definition of a limit mapping,

$$P_\beta \circ \varprojlim \exp(\Phi)\big(H(F)\big) = \varphi_\beta\big(\pi_{\varphi(\beta)}(F)\big), \quad \beta \in B,$$

where P_β is the natural projection of $X\{\exp(Y_\beta) : \beta \in B\}$ onto $\exp(Y_\beta)$. On the other hand, let us prove first that

$$\rho_\beta \circ \exp(\varprojlim \Phi)(F) = \varphi_\beta\big((\pi_{\varphi(\beta)}(F)\big),$$

where ρ_β is the natural projection of $X\{Y_\rho : \beta \in B\}$ onto Y_β. Indeed,

$$P_\beta \circ \exp(\varprojlim \Phi)\ (F) = \rho_\beta\big(\bigcup\{\varprojlim \Phi(x) : x \in F\}\big)$$

$$= \bigcup\{\rho_\beta \circ \varprojlim \Phi(x) : x \in F\} = \bigcup\{\varphi_\beta \circ \pi_{\varphi(\beta)}(x) : x \in F\}$$

$$= \varphi_\beta\big(\bigcup\{\pi_{\varphi(\beta)}(x) : x \in F\} = \varphi_\beta\big(\pi_{\varphi(\beta)}(F)\big).$$

Hence

$$P_\beta \circ K \circ \exp(\varprojlim \Phi)\ (F) = \rho_\beta\big(\exp(\varprojlim \Phi)\ (F)\big) = \varphi_\beta\big(\pi_{\varphi(\beta)}(F)\big).$$

Thus the commutativity of the diagram has been proved.

Note that 2.2 and 2.3, when taken together, mean that two functors exp and inverse limit commute.

2.4. *For any $f : X \to Y$ in $\mathcal{K}$, the mapping $f^{(\omega)} : X^{(\omega)} \to Y^{(\omega)}$ is exponentially complete.*

P r o o f. Applying 2.3 to the mapping of the inverse systems $\{f^{(n)}\} : \{X^{(n)}, u^{(n)}\} \to \{Y^{(n)}, u^{(n)}\}$ we get

$$\begin{array}{ccccc}
\exp(X^{(\omega)}) & \xrightarrow{\ H\ } & \varprojlim\{X^{(n+1)}, u^{(n+1)}\} & \xrightarrow{\ h\ } & X^{(\omega)} \\
\downarrow{\scriptstyle \exp(f^{(\omega)})} & & \downarrow{\scriptstyle \varprojlim\{f^{(n+1)}\}} & & \downarrow{\scriptstyle f^{(\omega)}} \\
\exp(Y^{(\omega)}) & \xrightarrow[\ K\ ]{} & \varprojlim\{Y^{(n+1)}, u^{(n+1)}\} & \xrightarrow[\ k\ ]{} & Y^{(\omega)}
\end{array}$$

h and k being the obvious homeomorphisms. Since the rectangles are commutative we obtain

$$\exp(f^{(\omega)}) = (k \circ K)^{-1} \circ f^{(\omega)} \circ (h \circ H).$$

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