PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 13 (27) 1972, pp. 65—76

# ON $GD$-GROUPOIDS WITH APPLICATIONS TO $n$-ARY QUASIGROUPS

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(Communicated November 26, 1971)

**1.** Let $S, T, V$, be three non empty sets and

$$A: S \times T \rightarrow V,$$

then we call the ordered quadruple $(S, T, V; A)$ a $G$-groupoid (a generalized groupoid).

Let us introduce the following denotations

$$L_a^A x = A(a, x), \quad R_b^A x = A(x, b)$$

where $a \in S$, $b \in T$ are some fixed elements. We call the mappings $L_a^A$ and $R_b^A$ the left and right $G$-translation with respect to $a$ and $b$.

If $L_a^A$ and $R_b^A$ are surjections, then we call the $G$-groupoid $A$ a $GD$-groupoid (a $G$-groupoid with a division). Namely, we call a $G$-groupoid $A$ a $GD$-groupoid if for every $a \in S$, $b \in T$, $c \in V$ the equations

$$A(x, b) = c \quad \text{and} \quad A(a, y) = c$$

always have solutions in $x \in S$ and $y \in T$, but not necessary unique ones. If those solutions are unique (or if $L_a^A$ and $R_b^A$ are bijections) then the $G$-groupoid $A$ we call a $G$-quasigroup.

Now, we shall introduce the notion of homotopy for $G$-and $GD$-groupoids and $G$-quasigroups.

Definition. *A $G$-groupoid $(S', T', V'; B)$ is a homotopy image of a $G$-groupoid $(S, T, V; A)$ if there exists a triple $H = [\alpha, \beta, \gamma]$ of surjections $\alpha: S \rightarrow S'$, $\beta: T \rightarrow T'$, $\gamma: V \rightarrow V'$, so that the following relation*

$$\gamma A(x, y) = B(\alpha x, \beta y)$$

*holds for every $x \in S$, $y \in T$.*

If $\alpha, \beta, \gamma$ are bijections then a homotopy is an isotopy.

Lemma. *A homotopy image of a $GD$-groupoid is also a $GD$-groupoid.*

P r o o f. Let $(S', T', V'; B)$ be a homotopy image of a $GD$-groupoid $(S, T, V; A)$ under the homotopy $H=[\alpha, \beta, \gamma]$. Let us consider the equation $B(x', b')=c'$, where $b'\in T'$, $c'\in V'$ are some fixed elements. Let be $b'=\beta b$, $c'=\gamma c$, for $b\in T$, $c\in V$, The equation $A(x, b)=c$ has a set of solutions in $S$. Then by the homotopy $H$ we have

$$\gamma A(x, b)=B(\alpha x, \beta b)=\gamma c$$

i.e.

$$B(\alpha x, b')=c'.$$

Solutions of the last equation are $\alpha x$ from $S$, for every $x$ such that $A(x, b)=c$ holds.

The existence of solutions of the equation $B(a', y')=c'$ is to be proved in a similar manner.

**2.** In this section we prove a theorem that will be a generalization of Belousov's theorem of four quasigroups [1], and then give a generalization of it.

T h e o r e m 1. *If four GD-groupoids $A, B, C, D$ where*

$$B: S_1\times S_2\to S_4 \qquad D: S_2\times S_3\to S_5$$

$$A: S_4\times S_3\to S \qquad C: S_1\times S_5\to S$$

*satisfy the equation*

$$A(B(x, y), z)=C(x, D(y, z)) \tag{1}$$

*for every $x\in S_1$, $y\in S_2$, $z\in S_3$ and if $L_a^C$ and $R_c^A$ are bijections for fixed $a\in S_1$ and $c\in S_3$, then there is a group $(S, \circ)$ which is a homotopy image for all GD-groupoids $A, B, C, D$.*

P r o o f. By fixing in (1), one after another, $x=a\in S_1$, $y=b\in S_2$, $z=c\in S_3$. then $x=a$ and $z=c$, we have

$$\begin{aligned} &A(L_a^B y, z)=L_a^C D(y, z)\\ &A(R_b^B x, z)=C(x, L_b^D z)\\ &R_c^A B(x, y)=C(x, R_c^D y)\\ &R_c^A L_a^B=L_a^C R_c^D. \end{aligned} \tag{2}$$

Let us introduce the equality

$$A(u, z)=R_c^A u\circ L_a^C L_b^D z. \tag{3}$$

This equality, because of assumptions of $A$, defines at least one operation "$\circ$" on the set $S$. In that purpose we prove that for arbitrary two elements $s, t\in S$ is defined one element $s\circ t\in S$. For the element $s$ there is only one element $u\in S_4$ such that $s=R_c^A u$, because $R_c^A$ is a bijection. On the other side, for $t$ there can exist $z_1, z_2\in S_3$ such that

$$t=L_a^C L_b^D z_1=L_a^C L_b^D z_2 \tag{4}$$

holds, i.e.

$$s \circ t = R_c^A u \circ L_a^C L_b^D z_1 = A(u, z_1)$$

$$s \circ t = R_c^A u \circ L_a^C L_b^D z_2 = A(u, z_2).$$

Let us prove that $A(u, z_1) = A(u, z_2)$.

From (4) under assumptions of $L_a^C$ we obtain $L_b^D z_1 = L_b^D z_2$, hence

$$L_x^C L_b^D z_1 = L_x^C L_b^D z_2. \tag{5}$$

As for $b \in S_2$ and an arbitrary $u \in S_4$ there is an $x \in S_1$ such that $B(x, b) = u$, then from (5) and (1) we have

$$C(x, D(b, z_1)) = C(x, D(b, z_2))$$

i.e.

$$A(B(x, b), z_1) = A(B(x, b), z_2)$$

respectively

$$A(u, z_1) = A(u, z_2).$$

In such a way we have proved that the operation "$\circ$" is well-defined on the set $S$. Thus $(S, S, S; \circ) \overset{\text{def}}{=} (S, \circ)$ is a homotopy image of the $GD$-groupoid $(S_4, S_3, S; A)$ under the homotopy $H = [R_c^A, L_a^C L_b^D, 1]$ ($1$ = the identity mapping of $S$). From Lemma it follows that $(S, \circ)$ is a groupoid with division. From the proof that "$\circ$" is well-defined, it also follows that $(S, \circ)$ is a quasigroup.

From (2) and (3) we have

$$\begin{aligned} A(u, z) &= R_c^A u \circ L_a^C L_b^D z \\ B(x, y) &= (R_c^A)^{-1} (R_c^A R_b^B x \circ L_a^C R_c^D y) \\ C(x, v) &= R_c^A R_b^B x \circ L_a^C v \\ D(y, z) &= (L_a^C)^{-1} (R_c^A R_a^B y \circ L_a^C L_b^D z). \end{aligned} \tag{6}$$

Hence, it follows that the quasigroup $(S, \circ)$ is a homotopic image for all $GD$-groupoids $A, B, C, D$. To prove that $(S, \circ)$ is a group, it is sufficient that the operation "$\circ$" is associative on $S$.

By substitution of (6) in (1) we have

$$(R_c^B R_b^A x \circ L_a^C L_c^D y) \circ L_a^C L_b^D z = R_c^A R_b^B x \circ (R_c^A L_a^B y \circ L_a^C L_b^D z) \tag{7}$$

and, as $L_a^C R_c^D = R_c^A L_a^B$, then (7) is reduced to

$$(\xi \circ \eta) \circ \zeta = \xi \circ (\eta \circ \zeta)$$

for every $\xi, \eta, \zeta, \in S$. Thus $(S, \circ)$ is a group.

R e m a r k. The cardinal numbers of the sets $S_4$, $S_5$ and $S$ are the same, for $R_c^A$ and $L_a^C$ are bijections.

Example. Let $S_1=\{x_1, x_2, x_3, x_4\}$, $S_2=\{y_1, y_2, y_3\}$, $S_3=\{z_1, z_2, z_3, z_4, z_5,\}$, $S_4=\{a, b, c\}$, $S_5=\{p, q, r\}$ $S=\{u, v, w\}$ be given sets, and the mappings $A, B, C, D$ be defined by following tables

| $B$ | $y_1$ | $y_2$ | $y_3$ |
|---|---|---|---|
| $x_1$ | $a$ | $b$ | $c$ |
| $x_2$ | $c$ | $a$ | $b$ |
| $x_3$ | $b$ | $c$ | $a$ |
| $x_4$ | $c$ | $a$ | $b$ |

| $D$ | $z_1$ | $z_2$ | $z_3$ | $z_4$ | $z_5$ |
|---|---|---|---|---|---|
| $y_1$ | $p$ | $r$ | $q$ | $p$ | $q$ |
| $y_2$ | $q$ | $p$ | $r$ | $q$ | $r$ |
| $y_3$ | $r$ | $q$ | $p$ | $r$ | $p$ |

| $A$ | $z_1$ | $z_2$ | $z_3$ | $z_4$ | $z_5$ |
|---|---|---|---|---|---|
| $a$ | $u$ | $v$ | $w$ | $u$ | $w$ |
| $b$ | $w$ | $u$ | $v$ | $w$ | $v$ |
| $c$ | $v$ | $w$ | $u$ | $v$ | $u$ |

| $C$ | $p$ | $q$ | $r$ |
|---|---|---|---|
| $x_1$ | $u$ | $w$ | $v$ |
| $x_2$ | $v$ | $u$ | $w$ |
| $x_3$ | $w$ | $v$ | $u$ |
| $x_4$ | $v$ | $u$ | $w$ |

It is easy to examine that $GD$-groupoids $A, B, C, D$ satisfy the equality

$$A(B(x, y), z)=C(x, D(y, z))$$

for every $x\in S_1$, $y\in S_2$, $z\in S_3$, and also $R_x^A$, $L_z^C$ are bijections, for some fixed $x\in S_1$, $z\in S_3$. Also, one can easily show that all $GD$-groupoids $A, B, C, D$ are homotopic to the group $(S, \circ)$, where the operation "$\circ$" is defined by the table

| $\circ$ | $u$ | $v$ | $w$ |
|---|---|---|---|
| $u$ | $u$ | $v$ | $w$ |
| $v$ | $v$ | $w$ | $u$ |
| $w$ | $w$ | $u$ | $v$ |

The above table is obtained from

$$A(x, y)=R_{z_1}^A x \circ L_{x_1}^C L_{y_1}^D y$$

for fixed elements $x_1, y_1, z_1$.

Another choice of fixed elements will give an isomorphic group to the group $(S, \circ)$.

Let us introduce an equivalence relation in the set of all surjections from a set $P$ onto a set $Q$ in the following manner

$$\alpha\sim\beta \overset{\text{def}}{\Leftrightarrow} (\exists\, a, b\in Q)(\forall\, x\in P)(\alpha x=a\cdot\beta x\cdot b)$$

where $(Q, \cdot)$ is a group.

If we put in (6)

$$\alpha=R_c^A \quad \beta=L_a^C L_b^D, \quad \gamma=R_c^A R_b^B, \quad \delta=L_a^C R_c^D, \quad \varphi=L_a^C$$

then, using the equality $R_c^A L_a^B=L_a^C R_c^D$, we have.

Theorem 2. *If four GD-groupoids $A, B, C, D$ satisfy the conditions of Theorem 1. then the general solution of the equation* (1) *is*

$$A(x, y) = \alpha x \circ \beta y$$

$$B(x, y) = \alpha^{-1}(\gamma x \circ \delta y)$$

$$C(x, y) = \gamma x \circ \varphi y$$

$$D(x, y) = \varphi^{-1}(\delta x \circ \beta y)$$

*where "$\circ$" is a group determined up to the isomorphism, and the mapping $\alpha, \beta, \gamma, \delta, \varphi$ up to the equivalence.*

Corollary. *If $S_1 = S_2 = S_3 = S_4 = S_5 = S$ and $\alpha, \beta, \gamma, \delta, \varphi$ are permutations of the set $S$, then we obtain Belousov's theorem on four quasigroups* ([1] *pp.* 93).

**3.** In the paper [2] the following functional equation of general associativity is considered

$$(8) \quad \begin{aligned} &A(x_1, x_2, \ldots, x_{i-1}, B(x_i, x_{i+1}, \ldots, x_{i+m-1}), x_{i+m}, \ldots x_p) \\ &= C(x_1, x_2, \ldots, x_{j-1}, D(x_j, x_{j+1}, \ldots, x_{j+n-1}), x_{j+n}, \ldots, x_p) \end{aligned}$$

where $A, B, C, D$ are quasigroups defined on the same non empty set $Q$ of the arity

$$|A| = p - m + 1, \quad |B| = m, \quad |C| = p - n + 1, \quad |D| = n$$

(The arity of an operation $K$ is designated by $|K|$). A special case of the equation (8) for $i = 1$, $j + n - 1 = p$ is considered by M. Hosszú [3].

Now we shall show that some results from [2] can be obtained by application of Theorem 2.

Let us introduce some abbreviations. We designate a sequence $x_k, x_{k+1}, \ldots, x_r$ by $x_k^r$. If $r < k$, then the symbol $x_k^r$ is empty. In the same manner an ordered $r - k + 1$ – tiple $(x_k, x_{k+1}, \ldots, x_r)$ is designated by $(x_k^r)$.

By using the introduced designations the equation (8) is reduced to the form

$$(9) \qquad A(x_1^{i-1}, \ B(x_i^{i+m-1}), \ x_{i+m}^p) = C(x_1^{j-1}, \ D(x_j^{j+n-1}), \ x_{j+n}^p).$$

Let us consider the equation (9) for the case that $i \leqslant j$, $j < i + m \leqslant j + n$.

We fix in (9) common variables for the operations $A$, $C$, i.e. we fix $x_1^{i-1}$ by $a_1^{i-1}$ and $x_{j+n}^p$ by $a_{j+n}^p$. Then (9) is reduced to

$$(10) \qquad A_1(B(x_i^{j-1}, \ x_j^{i+m-1}), \ x_{i+m}^{j+n-1}) = C_1(x_i^{j-1}, \ D(x_j^{i+m-1}, \ x_{i+m}^{j+n-1}))$$

where

$$A_1(x, x_{i+m}^{j+n-1}) = A(a_1^{i-1}, \ x, x_{i+m}^{j+n-1}, \ a_{j+n}^p)$$

$$C_1(x_i^{j-1}, x) = C(a_1^{i-1}, \ x_i^{j-1}, \ x, a_{j+n}^p)$$

and $A_1$, $C_1$ are quasigroups of the arity $|A_1| = j + n - i - m + 1$, $|C_1| = j - i + 1$.

Let us introduce

$$(11)\quad \begin{aligned} A_1\left(x, x_{i+m}^{j+n-1}\right) &\stackrel{\text{def}}{=} \tilde{A}_1\left(x, \left(x_{i+m}^{j+n-1}\right)\right) \\ B\left(x_i^{j-1}, x_j^{i+m-1}\right) &\stackrel{\text{def}}{=} \tilde{B}\left(\left(x_i^{j-1}\right), \left(x_j^{i+m-1}\right)\right) \\ C_1\left(x_i^{j-1}, x\right) &\stackrel{\text{def}}{=} \tilde{C}_1\left(\left(x_i^{j-1}\right), x\right) \\ D\left(x_j^{i+m-1}, x_{i+m}^{j+n-1}\right) &\stackrel{\text{def}}{=} \tilde{D}_1\left(\left(x_j^{i+m-1}\right), \left(x_{i+m}^{j+n-1}\right)\right). \end{aligned}$$

If we still introduce designations $X=\left(x_i^{j-1}\right)$, $Y=\left(x_j^{i+m-1}\right)$, $Z=\left(x_{i+m}^{j+n-1}\right)$ then (10) is reduced to

$$(12)\quad \tilde{A}_1\left(\tilde{B}(X, Y), Z\right)=\tilde{C}_1\left(X, \tilde{D}(Y, Z)\right)$$

where $\tilde{A}_1$, $\tilde{B}$, $\tilde{C}_1$, $\tilde{D}$, because of (11), are $GD$-groupoids. As

$$\tilde{B}: Q^{j-i}\times Q^{i+m-j}\rightarrow Q \qquad \tilde{D}: Q^{i+m-j}\times Q^{j+n-i-m}\rightarrow Q$$

$$\tilde{A}_1: Q\times Q^{j+n-i-m}\rightarrow Q \qquad \tilde{C}_1: Q^{j-i}\times Q\rightarrow Q$$

and because of the relations (11), by means of which these $GD$-groupoids are defined, we have that $GD$-groupoids $\tilde{A}_1$, $\tilde{B}$, $\tilde{C}_1$, $\tilde{D}$ satisfy the conditions of Theorem 2. Then from (12) we have

$$\tilde{B}(X, Y)=\alpha^{-1}(\gamma X\circ\delta Y), \quad \tilde{D}(Y, Z)=\varphi^{-1}(\delta Y\circ\beta Z)$$

i.e.

$$(13)\quad \begin{aligned} B\left(x_i^{j-1}, x_j^{i+m-1}\right) &=\alpha^{-1}\left(\gamma\left(x_i^{j-1}\right)\circ\delta\left(x_j^{i+m-1}\right)\right) \\ D\left(x_j^{i+m-1}, x_{i+m}^{j+n-1}\right) &=\varphi^{-1}\left(\delta\left(x_j^{i+m-1}\right)\circ\beta\left(x_{i+m}^{j+n-1}\right)\right) \end{aligned}$$

where $(Q, \circ)$ is a group, $\gamma$, $\delta$, $\beta$ are quasigroups of the arity $|\gamma|=j-i$, $|\delta|=i+m-j$, $|\beta|=j+n-i-m$, and $\alpha$, $\varphi$ are permutations of the set $Q$.

By substitution (13) into (9) we obtain

$$(14)\quad \begin{aligned} & A\left(x_1^{i-1}, \alpha^{-1}\left(\gamma\left(x_i^{j-1}\right)\circ\delta\left(x_j^{i+m-1}\right)\right), x_{i+m}^{p}\right) \\ & =C\left(x_1^{j-1}, \varphi^{-1}\left(\delta\left(x_j^{i+m-1}\right)\circ\beta\left(x_{i+m}^{j+n-1}\right)\right), x_{j+n}^{p}\right) \end{aligned}$$

Hence, by fixing $x_i^{j-1}$ by $c_i^{j-1}$ in such a way that $\gamma\left(c_i^{j-1}\right)=e$ ($e$ = identity elements of the group "$\circ$"), we have

$$(15)\quad \begin{aligned} & A\left(x_1^{i-1}, \alpha^{-1}\delta\left(x_j^{i+m-1}\right), x_{i+m}^{p}\right) \\ & =K\left(x_1^{i-1}, \delta\left(x_j^{i+m-1}\right)\circ\beta\left(x_{i+m}^{j+n-1}\right), x_{j+n}^{p}\right) \end{aligned}$$

where

$$K\left(x_1^{i-1}, y, x_{j+n}^{p}\right)=C\left(x_1^{i-1}, c_i^{j-1}, \varphi^{-1}y, x_{j+n}^{p}\right)$$

is a quasigroup of the arity $|K|=i+p-j-n$.

If we put $\alpha^{-1}\beta(x_j^{i+m-1})=x$ then from (15) we obtain

$$A(x_1^{i-1},\ x,\ x_{i+m}^p)=K(x_1^{i-1},\ \alpha x\circ\beta(x_{i+m}^{j+n-1}),\ x_{j+n}^p). \tag{16}$$

If we put $\delta(x_j^{i+m-1})=e$ into (14), then

$$C(x_1^{j-1},\ \varphi^{-1}\beta(x_{i+m}^{j+n-1}),\ x_{j+n}^p)=A(x_1^{i-1},\ \alpha^{-1}\gamma(x_i^{j-1}),\ x_{i+m}^p)$$

Hence, because of (15) we have

$$C(x_1^{j-1},\ \varphi^{-1}\beta(x_{i+m}^{j+n-1}),\ x_{j+n}^p)=K(x_1^{i-1},\ \gamma(x_i^{j-1})\circ\beta(x_{i+m}^{j+n-1}),\ x_{j+n}^p)$$

or, if we put $\varphi^{-1}\beta(x_{i+m}^{j+n-1})=y$, then

$$C(x_1^{j-1},\ y,\ x_{j+n}^p)=K(x_1^{i-1},\ \gamma(x_i^{j-1})\circ\varphi y,\ x_{j+n}^p). \tag{17}$$

Finally, relations (13), (16) and (17) give

$$\begin{aligned} A(x_1^{i-1},\ x,\ x_{i+m}^p)&=K(x_1^{i-1},\ \alpha x\circ\beta(x_{i+m}^{j+n-1}),\ x_{j+n}^p)\\ B(x_i^{i+m-1})&=\alpha^{-1}(\gamma(x_i^{j-1})\circ\delta(x_j^{i+m-1}))\\ C(x_1^{j-1},\ y,\ x_{j+n}^p)&=K(x_1^{i-1},\ \gamma(x_i^{j-1})\circ\varphi y,\ x_{j+n}^p)\\ D(x_j^{j+n-1})&=\varphi^{-1}(\delta(x_j^{i+m-1})\circ\beta(x_{i+m}^{j+n-1})) \end{aligned} \tag{18}$$

representing the general solution of the equation (9).

Specially, for $i=j$, $n=m$, from (18) we have

$$D(x_i^{i+m-1})=\theta B(x_i^{i+m-1})$$

$$A(x_1^{i-1},\ x,\ x_{i+m}^p)=C(x_1^{i-1},\ \theta x,\ x_{i+m}^p)$$

where $\theta=\varphi^{-1}\alpha$ is a permutation of the set $Q$.

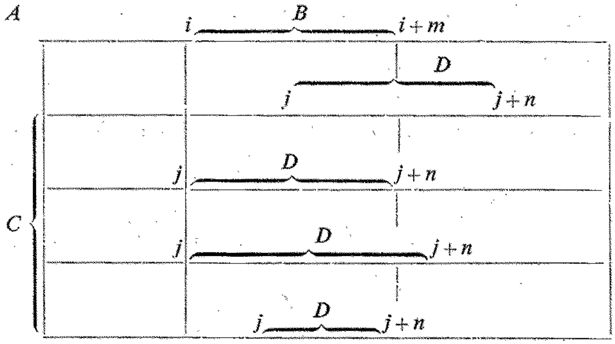
For $i=j$, $j<i+m<i+n$ from (18) we have

$$\begin{aligned} A(x_1^{i-1},\ x,\ x_{i+m}^p)&=K(x_1^{i-1},\ \alpha x\circ\beta(x_{i+m}^{i+n-1}),\ x_{i+n}^p)\\ B(x_i^{i+m-1})&=\alpha^{-1}\delta(x_i^{i+m-1})\\ C(x_1^{i-1},\ y,\ x_{i+n}^p)&=K(x_1^{i-1},\ \varphi y,\ x_{i+n}^p)\\ D(x_i^{i+n-1})&=\varphi^{-1}(\delta(x_i^{i+m-1})\circ\beta(x_{i+m}^{i+n-1})) \end{aligned}$$

For $i<j$, $i+m=j+n$ from (18) we have

$$\begin{aligned} A(x_1^{i-1},\ x,\ x_{i+m}^p)&=K(x_1^{i-1},\ \alpha x,\ x_{i+m}^p)\\ B(x_i^{i+m-1})&=\alpha^{-1}(\gamma(x_i^{j-1})\circ\delta(x_j^{i+m-1}))\\ C(x_1^{j-1},\ y,\ x_{i+m}^p)&=K(x_1^{i-1},\ \gamma(x_i^{j-1})\circ\varphi y,\ x_{i+m}^p)\\ D(x_j^{i+m-1})&=\varphi^{-1}\delta(x_j^{i+m-1}) \end{aligned}$$

The following scheme gives the conditions under which the equation (9) is solved




In other cases solutions of (9) are not expressed by means of one group operation and quasigroups of smaller length (see [2]), The solution for $i<j$, $i+m=j+n$ cannot be obtained from Theorem 2.1. from [2].

Now we shall give a generalization of Theorem 1.

T h e o r e m 3. *If GD-groupoids* $A_i$, $B_i$ $(i=1, \ldots, n)$, *where*

$$\begin{array}{ll} A_n : X_1 \times X_2 \to Q_n & B_n : X_n \times X_{n+1} \to P_n \\ A_{n-1} : Q_n \times X_3 \to Q_{n-1} & B_{n-1} : X_{n-1} \times P_n \to P_{n-1} \\ A_{n-2} : Q_{n-1} \times X_4 \to Q_{n-2} & B_{n-2} : X_{n-2} \times P_{n-1} \to P_{n-2} \\ \vdots & \vdots \\ A_2 : Q_3 \times X_n \to Q_2 & B_2 : X_2 \times P_3 \to P_2 \\ A_1 : Q_2 \times X_{n+1} \to Q & B_1 : X_1 \times P_2 \to Q \end{array}$$

*satisfy the equation*

(19) $$A_1 A_2 \cdots A_n x_1 x_2 \cdots x_{n+1} = B_1 x_1 B_2 x_2 \cdots B_n x_n x_{n+1}$$

*for every* $x_i \in X_i$ $(i=1, \ldots, n+1)$, *and if* $\alpha_k$, $\gamma_k$ $(k=1, \ldots, n-1)$ *are bijections defined*

$$\alpha_k u = A_k(u, a_k), \quad \gamma_k v = B_k(b_k, v)$$

*for any fixed* $a_k \in X_{n-k+2}$ *and* $bk \in X_k$, *then there is a quasigroup* $(Q, \circ)$ *being homotopic image for all GD-groupoids* $A_i$, $B_i$ $(i=1, \ldots, n)$, *and for which is valid*

$$(\cdots((u_1 \circ u_2) \circ u_3) \circ \cdots) \circ u_n) \circ u_{n+1} = u_1 \circ (u_2 \circ (u_3 \circ (\cdots \circ (u_n \circ u_{n+1}) \cdots).$$

P r o o f. Let us introduce the following abbreviations

$$\alpha_k^s \overset{\text{def}}{=} \alpha_s \alpha_{s+1} \cdots \alpha_k, \quad \gamma_k^s \overset{\text{def}}{=} \gamma_s \gamma_{s+1} \cdots \gamma_k$$

where $s \leqslant k$, $k=1, \ldots, n-1$, and the right side of the above equalities is a product of mappings. Clearly, the mapping $\alpha_k^s$ and $\gamma_k^s$ for every $k \in \{1, \ldots, n-1\}$ and every $s$, are bijections.

Let us designate by $\alpha_n, \gamma_n, \beta_k, \delta_k\ (k=1, \dots, n)$ mappings defined as follows

$$\alpha_n u = A_n(u, x_0) \qquad \gamma_n v = B_n(y_0, v)$$

$$\beta_k x = A_k(u_k, x) \qquad \delta_k y = B_k(y, v_k)$$

for any fixed $x_0 \in X_2$, $y_0 \in X_n$, $u_k \in Q_{k+1}$, $v_k \in P_{k+1}$. As $A_i$, $B_i$ are *GD*-groupoids, then the mappings $\alpha_n, \gamma_n, \beta_k, \delta_k$ are surjections.

By fixing in (19), in an order, $x_1 \in X_1$, $x_2 \in X_2, \dots$ $x_{n-1} \in X_{n-1}$ i.e. all $x$-es except $x_n$ and $x_{n+1}$; then all ones except $x_{n-1}$ and $x_{n+1}$, and so on; finally all ones except $x_1$ and $x_{n+1}$, we obtain

$$
(20) \qquad
\begin{aligned}
A_1(\beta_2 x_n, x_{n+1}) &= \gamma^1_{n-1} B_n(x_n, x_{n+1}) \\
A_1(\alpha_2 \beta_3 x_{n-1}, x_{n+1}) &= \gamma^1_{n-2} B_{n-1}(x_{n-1}, \gamma_n x_{n+1}) \\
A_1(\alpha^2_3 \beta_4 x_{n-2}, x_{n+1}) &= \gamma^1_{n-3} B_{n-2}(x_{n-2}, \gamma_{n-1}\gamma_n x_{n+1}) \\
A_1(\alpha^2_4 \beta_5 x_{n-3}, x_{n+1}) &= \gamma^1_{n-4} B_{n-3}(x_{n-3}, \gamma^{n-2}_{n-1}\gamma_n x_{n+1}) \\
&\ \ \vdots \\
A_1(\alpha^2_{n-1} \beta_n x_2, x_{n+1}) &= \gamma_1 B_2(x_2, \gamma^3_{n-1}\gamma_n x_{n+1}) \\
A_1(\alpha^2_{n-1} \alpha_n x_1, x_{n+1}) &= B_1(x_1, \gamma^2_{n-1}\gamma_n x_{n+1}).
\end{aligned}
$$

If now in (19), in the same manner, we fix all $x$-es except $x_1$ and $x_2$, then all ones except $x_1$ and $x_3$, and so on; finally all ones, except $x_1$ and $x_n$, we obtain

$$
(21) \qquad
\begin{aligned}
\alpha^1_{n-1} A_n(x_1, x_2) &= B_1(x_1, \delta_2 x_2) \\
\alpha^1_{n-2} A_{n-1}(\alpha_n x_1, x_3) &= B_1(x_1, \gamma_2 \delta_3 x_3) \\
\alpha^1_{n-3} A_{n-2}(\alpha_{n-1}\alpha_n x_1, x_4) &= B_1(x_1, \gamma^2_3 \delta_4 x_4) \\
\alpha^1_{n-4} A_{n-3}(\alpha^{n-2}_{n-1}\alpha_n x_1, x_5) &= B_1(x_1, \gamma^2_4 \delta_5 x_5) \\
&\ \ \vdots \\
\alpha^1_2 A_3(\alpha^4_{n-1}\alpha_n x_1, x_{n-1}) &= B_1(x_1, \gamma^2_{n-2} \delta_{n-1} x_{n-1}) \\
\alpha_1 A_2(\alpha^3_{n-1}\alpha_n x_1, x_n) &= B_1(x_1, \gamma^2_{n-1} \delta_n x_n)
\end{aligned}
$$

By fixing $x_{n+1} \in X_{n+1}$ in all equalities from (20), we obtain

$$\begin{aligned}
&\alpha_1 \beta_2 = \gamma_{n-1}^1 \delta_n \\
&\alpha_2^1 \beta_3 = \gamma_{n-2}^1 \delta_{n-1} \\
&\alpha_3^1 \beta_4 = \gamma_{n-3}^1 \delta_{n-2} \\
&\quad\vdots \\
&\alpha_{n-1}^1 \beta_n = \gamma_1 \delta_2 \\
&\alpha_{n-1}^1 \alpha_n = \delta_1 .
\end{aligned} \tag{22}$$

Let us consider the equality

$$A_1(u, x_{n+1}) = \alpha_1 u \circ \gamma_{n-1}^1 \gamma_n x_{n+1} . \tag{23}$$

That the equality (23) defines at least one operation "$\circ$" on the set $Q$ is to be proved in the same manner as the equality (3) of Theorem 1. Thus, the quasigroup $(Q, \circ)$ because of (23) is a homotopic image of $GD$-groupoid $(Q_2, X_{n+1}, Q; A_1)$.

From the equality (20) taking (23) we have

$$B_1(x_1, \gamma_{n-1}^2 \gamma_n x_{n+1}) = A_1(\alpha_{n-1}^2 \alpha_n x_1, x_{n+1}) = \alpha_{n-1}^1 \alpha_n x_1 \circ \gamma_{n-1}^1 \gamma_n x_{n+1}$$

or if we put $\gamma_{n-1}^2 \gamma_n x_{n+1} = v \in P_2$ then

$$B_1(x_1, v) = \alpha_{n-1}^1 \alpha_n x_1 \circ \gamma_1 v. \tag{24}$$

From (20) and (21) taking (23) and (24) we obtain

$$\begin{aligned}
A_1(x, y) &= \alpha_1 x \circ \gamma_{n-1}^1 \gamma_n y \\
A_2(x, y) &= \alpha_1^{-1} (\alpha_2^1 x \circ \gamma_{n-1}^1 \delta_n y) \\
A_3(x, y) &= (\alpha_2^1)^{-1} (\alpha_3^1 x \circ \gamma_{n-2}^1 \delta_{n-1} y) \\
&\ \ \vdots \\
A_{n-1}(x, y) &= (\alpha_{n-2}^1)^{-1} (\alpha_{n-1}^1 x \circ \gamma_2^1 \delta_3 y) \\
A_n(x, y) &= (\alpha_{n-1}^1)^{-1} (\alpha_{n-1}^1 \alpha_n x \circ \gamma_1 \delta_2 y) \\
B_1(x, y) &= \alpha_{n-1}^1 \alpha_n x \circ \gamma_1 y \\
B_2(x, y) &= \gamma_1^{-1} (\alpha_{n-1}^1 \beta_n x \circ \gamma_2^1 y) \\
&\ \ \vdots \\
B_{n-3}(x, y) &= (\gamma_{n-4}^1)^{-1} (\alpha_4^1 \beta_5 x \circ \gamma_{n-3}^1 y) \\
B_{n-2}(x, y) &= (\gamma_{n-3}^1)^{-1} (\alpha_3^1 \beta_4 x \circ \gamma_{n-2}^1 y) \\
B_{n-1}(x, y) &= (\gamma_{n-2}^1)^{-1} (\alpha_2^1 \beta_3 x \circ \gamma_{n-1}^1 y) \\
B_n(x, y) &= (\gamma_{n-1}^1)^{-1} (\alpha_1 \beta_2 x \circ \gamma_{n-1}^1 y).
\end{aligned} \tag{25}$$

From here, we have that the quasigroup $(Q, \circ)$ is the homotopic image for all $GD$-groupoids $A_i$, $B_i$, $(i=1, \ldots, n)$.

By substitution (25) into (19) we obtain

$$(\cdots(\alpha_{n-1}^1 \alpha_n x_1 \circ \gamma_1 \delta_2 x_2) \circ \gamma_2^1 \delta_3 x_3) \circ \cdots) \circ \gamma_{n-1}^1 \delta_n x_n) \circ \gamma_{n-1}^1 \gamma_n x_{n+1}$$
$$= \alpha_{n-1}^1 \alpha_n x_1 \circ (\alpha_{n-1}^1 \alpha_n x_2 \circ (\cdots \circ (\alpha_2^1 \beta_3 x_{n-1} \circ (\alpha_1 \beta_2 x_n \circ \gamma_{n-1}^1 \gamma_n x_{n+1}) \cdots)$$

hence, because of (22) we have

$$(\cdots(u_1 \circ u_2) \circ u_3) \circ \cdots) \circ u_n) \circ u_{n+1} = u_1 \circ (u_2 \circ (\cdots \circ (u_{n-1} \circ (u_n \circ u_{n+1}) \cdots)$$

Example. Let $(Q, A_i)$ and $(Q, B_i)$ $(i=1, 2, 3)$ be quasigroups satisfying the equation

$$(26) \qquad \begin{aligned} &A_1(A_2(A_3(x_1, \ldots, x_n), x_{n+1}, \ldots, x_m), x_{m+1}, \ldots, x_p) \\ &= B_1(x_1, \ldots, x_i, B_2(x_{i+1}, \ldots, x_n, B_3(x_{n+1}, \ldots, x_p))) \end{aligned}$$

and of arity

$$|A_1| = p-m+1 \quad |A_2| = m-n+1 \quad |A_3| = n$$
$$|B_1| = i+1 \qquad |B_2| = n-i+1 \quad |B_3| = p-n \qquad (1<i<n<m<p)$$

From Theorem 3. we have immediately the general solution of the equation (26). In that purpose we put

$$(27) \qquad \begin{aligned} A_1(u, x_{m+1}, \ldots, x_p) &\overset{\text{def}}{=} \tilde{A}_1(u, (x_{m+1}, \ldots, x_p)) \\ A_2(v, x_{n+1}, \ldots, x_m) &\overset{\text{def}}{=} \tilde{A}_2(v, (x_{n+1}, \ldots, x_m)) \\ A_3(x_1, \ldots, x_n) &\overset{\text{def}}{=} \tilde{A}_3((x_1, \ldots, x_i), (x_{i+1}, \ldots, x_n)) \\ B_1(x_1, \ldots, x_i, w) &\overset{\text{def}}{=} \tilde{B}_1((x_1, \ldots, x_i), w) \\ B_2(x_{i+1}, \ldots, x_n, t) &\overset{\text{def}}{=} \tilde{B}_2((x_{i+1}, \ldots x_n), t) \\ B_3(x_{n+1}, \ldots, x_p) &\overset{\text{def}}{=} \tilde{B}_3((x_{n+1}, \ldots, x_m), (x_{m+1}, \ldots, x_p)) \end{aligned}$$

where $(x_1, \ldots, x_i) \in Q^i$, $(x_{i+1}, \ldots, x_n) \in Q^{n-i}$, $(x_{n+1}, \ldots, x_m) \in Q^{m-n}$, $(x_{m+1}, \ldots, x_p) \in Q^{p-m}$, $u, v, w, t \in Q$.

Then (26) is reduced to

$$(28) \qquad \tilde{A}_1(\tilde{A}_2(\tilde{A}_3(z_1, z_2), z_3), z_4) = \tilde{B}_1(z_1, \tilde{B}_2(z_2, \tilde{B}_3(z_3, z_4)))$$

where

$$\tilde{A}_1 : Q \times Q^{p-m} \to Q \qquad \tilde{B}_1 : Q^i \times Q \to Q$$
$$\tilde{A}_2 : Q \times Q^{m-n} \to Q \qquad \tilde{B}_2 : Q^{n-i} \times Q \to Q$$
$$\tilde{A}_3 : Q^i \times Q^{n-i} \to Q \qquad \tilde{B}_3 : Q^{m-n} \times Q^{p-m} \to Q$$

are *GD*-groupoids for which the conditions of Theorem 3. are satisfied. It is easy to show from (27), using the assumption, that $A_i$ and $B_i$ $(i=1, 2, 3)$ being quasigroups, we have

$$\tilde{A}_1(x, y)=\alpha_1 x \circ \gamma_2^1\gamma_3 y \qquad \tilde{B}_1(x, y)=\alpha_2^1\alpha_3 x \circ \gamma_1 y$$
$$\tilde{A}_2(x, y)=\alpha_1^{-1}(\alpha_2^1 x \circ \gamma_2^1\delta_3 y) \qquad \tilde{B}_2(x, y)=\gamma_1^{-1}(\alpha_2^1\beta_3 x \circ \gamma_2^1 y)$$
$$\tilde{A}_3(x, y)=(\alpha_2^1)^{-1}(\alpha_2^1\alpha_3 x \circ \gamma_1\delta_2 y) \qquad \tilde{B}_3(x, y)=(\gamma_2^1)^{-1}(\alpha_1\beta_2 x \circ \gamma_2^1\gamma_3 y).$$

By using (22) and introducing new denotations

$$\alpha_1=\alpha,\ \gamma_2^1\gamma_3=\beta.\ \alpha_2^1=\gamma,\ \alpha_1\beta_2=\gamma_2^1\delta_3=\delta,\ \gamma_1=\eta,$$
$$\gamma_2^1=\xi,\ \alpha_2^1\alpha_3=\varphi,\ \alpha_2^1\beta_3=\gamma_1\delta_2=\psi$$

because of (27) we obtain

$$A_1(u, x_{m+1}, \ldots, x_p)=\alpha u \circ \beta(x_{m+1}, \ldots, x_p)$$
$$A_2(v, x_{n+1}, \ldots, x_m)=\alpha^{-1}(\gamma v \circ \delta(x_{n+1}, \ldots, x_m))$$
$$A_3(x_1, \ldots, x_n) \quad =\gamma^{-1}(\varphi(x_1, \ldots, x_i) \circ \psi(x_{i+1}, \ldots, x_n))$$
$$B_1(x_1, \ldots, x_i, w) \quad =\varphi(x_1, \ldots, x_i) \circ \eta w$$
$$B_2(x_{i+1}, \ldots, x_n, t)=\eta^{-1}(\psi(x_{i+1}, \ldots, x_n) \circ \xi t)$$
$$B_3(x_{n+1}, \ldots, x_p) \quad =\xi^{-1}(\delta(x_{n+1}, \ldots, x_m) \circ \beta(x_{m+1}, \ldots, x_p))$$

where $(Q, \circ)$ is a binary quasigroup for which the following condition is satisfied

$$((u_1 \circ u_2) \circ u_3) \circ u_4=u_1 \circ (u_2 \circ (u_3 \circ u_4)).$$

The mappings

$$\alpha,\ \gamma,\ \eta,\ \xi: Q \to Q$$

arr bijections, and $\beta$, $\delta$, $\varphi$ and $\psi$, where

$$\beta: Q^{p-m} \to Q, \quad \delta: Q^{m-n} \to Q, \quad \varphi: Q^i \to Q, \quad \psi: Q^{n-i} \to Q$$

are surjections, i.e. quasigroups of arity

$$|\beta|=p-m, \quad |\delta|=m-n, \quad |\varphi|=i, \quad |\psi|=n-i.$$

Remark. The quasigroup $(Q, \circ)$, being a homotopic image for all *GD*-groupoids from Theorem 3, is isotopic to a group [4].

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