PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 13 (27), 1972, pp. 59—63

ON THE FUNCTIONAL EQUATION $f=fg$

Ž. Mijajlović

(Received December 10, 1971)

1. Let A and B be any non-empty sets and g a given permutation of the set A. The problem to find all mappings $f: A \to B$ which satisfy the equation $f(x)=f(g(x))$ is already known. The general solution of the preceding equation was found by S. Prešić in [1]. Our aim is to show that it is possible to applicate the idea, used by Prešić, in solving the following problem:

Let K be a given category and A, B two objects of this category and let $g \in \mathrm{Aut}(A)$. Then determine all $f \in \mathrm{Mor}(A, B)$ such that

(1) $$f=fg.$$

Before beginning our study let us agree with some terminology (which) is, for example, given in [3]). The action of groupoid G on set S (from right) is mapping $S \times G \to S$, which is denoted by $*$, if fulfilled:

(i) $s \in S$ and $x, y \in G$ implies $s*(xy)=(s*x)*y$.

(ii) If e is the neutral element of the groupoid G, then $s*e=s$.

In this case we call S the G-set. For $s \in S$, the set $s*G=\{s*x/x \in G\}$ is called the orbit of the element s. The mapping $F: S \to S'$ (S and S' are G-sets), is called G mapping if for $s \in S$ and $x \in G$ is fulfilled

$$F(s*x)=F(s)*x$$

Example 1. The mapping $\mathrm{Mor}(A, B) \times \mathrm{Aut}(A) \to \mathrm{Mor}(A, B)$, denoted by $\circ$, is an action if we define $f \circ h = fh$ for $f \in \mathrm{Mor}(A, B)$ and $h \in \mathrm{Aut}(A)$. Really as

(i) $$f \circ (h_1 h_2) = f(h_1 h_2) = (f \circ h_1) \circ h_2$$

(ii) $$f \circ i_A = f i_A = f$$

we see that $\circ$ is an action. That means $\mathrm{Mor}(A, B)$ is $\mathrm{Aut}(A)$ – set. The orbit of the morphism f will be $f \circ \mathrm{Aut}(A) = \{fh/h \in \mathrm{Aut}(A)\}$.

We note that we shall use the following fact:

(2) If G is group, then, for all $s_1, s_2 \in S$, $s_1 * G$ and $s_2 * G$ are either disjunct or coincident. That means that the orbits determine a decomposition of the set S.

II. In this section we shall find the general solution of equation (1).

Let g be given permutation in (1). In our consideration G is a cyclic group generated by g, i.e. $G=\{g^n/n \in D\}$, where D denotes the set of integers. We shall denote the set of all solution of the equation (1) by $\mathcal{Y}_g$, i.e. $\mathcal{Y}_g=\{f/f=fg, f\in \text{Mor}(A,B)\}$. Then, we introduce the action, denoted by $\circ$, in the same way as in example 1., i.e. $f\circ h=fh$ for $f\in\text{Mor}(A,B)$ and $h\in G$. Further, we shall write fh instead of $f\circ h$. Let us denote the set of the orbits of morphisms f by $\mathcal{P}_g$. That means $\mathcal{P}_g=\{fG/f\in\text{Mor}(A,B)\}$. According to (2) $\mathcal{P}_g$ determines the relation of equivalence ρ within the set $\text{Mor}(A,B)$ ($f,f'\in\text{Mor}(A,B)$ implies $(f,f')\in\rho$ if and only if f and f' have the same orbits). Then it is evidently $\mathcal{P}_g=\text{Mor}(A,B)/\rho$. Notice that for the class of equivalence $\tilde{f}$ there is $\tilde{f}=fG=\{fg^n/n\in D\}$. Further on, let us introduce the action $\mathcal{P}_g xG\to\mathcal{P}_g$ as follows: for $\tilde{f}\in\mathcal{P}_g$ and $h\in G$ let $\tilde{f}*h=\{fg^n h/n\in D\}$. Mapping $*$ is an action for:

(i) $\tilde{f}*(h_1h_2)=\{fg^n(h_1h_2)/n\in D\}=\{(fg^nh_1)h_2/n\in D\}=(\tilde{f}*h_1)*h_2$.

(ii) $(\tilde{f}*i_A)=\{fg^n i_A/n\in D\}=\{fg^n/n\in D\}=\tilde{f}$.

Furthermore, for all $h\in G$ there is $\tilde{f}*h=\tilde{f}$, because $h=g^m$ for some $m\in D$, so $\tilde{f}*h=\{fg^{n+m}/n\in D\}=\{fg^k/k\in D\}=\tilde{f}$. According to the preceding $\text{Mor}(A,B)$ and $\mathcal{P}_g$ are G-sets.

The special roll in the future consideration will have the G-mapping:

$$\mathcal{M}:\mathcal{P}_g\to\text{Mor}(A,B)$$

which satisfies the following conditions:

(3) (i) If $\{f\}\in G$ then $\mathcal{M}(\{f\})=f$.
(ii) For $h\in G$, $\mathcal{M}(\tilde{f}*h)=\mathcal{M}(\tilde{f})h$ (the characteristic of G-mapping).

Let $\text{Im}\,\mathcal{M}$ be the value set of the mapping $\mathcal{M}$.

Lemma 1. $\mathcal{Y}_g=\text{Im}\,\mathcal{M}$.

Proof. If $f\in\text{Im}\,\mathcal{M}$, then there is $\tilde{h}\in\mathcal{P}_g$ such that $f=\mathcal{M}(\tilde{h})$. Further, $fg=\mathcal{M}(\tilde{h})g=\mathcal{M}(\tilde{h}*g)=\mathcal{M}(\tilde{h})=f$ i.e. $f=fg$. Conversely, let f satisfy the equation (1). Then $fg^n=f$ for all $n\in D$, and $\tilde{f}=\{f\}$. According to the characteristic (i) of the mapping $\mathcal{M}$, there is $\mathcal{M}(\tilde{f})=\mathcal{M}(\{f\})=f$. i.e. $f\in\text{Im}\,\mathcal{M}$.

Remark. If $\mathcal{M}$ and $\mathcal{M}'$ are two mappings which satisfy the condition (3), then $\text{Im}\,\mathcal{M}=\text{Im}\,\mathcal{M}'$.

Let p: $\text{Mor}(A,B)\to\mathcal{P}_g$ be cannonical mapping and $\Omega=\mathcal{M}p$ i.e. that the following diagram is commutative:

(4)
$$\begin{array}{ccc} \mathcal{P}_g & \xleftarrow{\;p\;} & \text{Mor}(A,B) \\ & \mathcal{M}\searrow \quad \swarrow\Omega & \\ & \text{Mor}(A,B) & \end{array}$$

Lemma 2. $\Omega\Omega=\Omega$.

Proof. We see that, for the orbit $\tilde{f}=\{fg^n/n\in D\}$ the following is true: $p\mathcal{M}(\tilde{f})=p\mathcal{M}(\{fg^n/n\in D\})=\mathcal{M}(\tilde{f})*G=\{\mathcal{M}(\tilde{f})g^n/n\in D\}=\{\mathcal{M}(\tilde{f}*g^n)/n\in D\}=$ $=\{\mathcal{M}(\tilde{f})\}$ because $n\in D$ implies $\tilde{f}*g^n=\tilde{f}$. Consequently, $\Omega\Omega(f)=(\mathcal{M}p)\times$ $\times(\mathcal{M}p)(f)=\mathcal{M}(p\mathcal{M})p(f)=\mathcal{M}(p\mathcal{M})(\tilde{f})=\mathcal{M}(\{\mathcal{M}(\tilde{f})\})=\mathcal{M}(\tilde{f})=\mathcal{M}p(f)=$ $=\Omega(f)$ i.e. $\Omega\Omega=\Omega$.

Lemma 3. $f \in \mathcal{Y}_g \Leftrightarrow \Omega(f)=f$.

Proof. Let $f=fg$. Then $p(f)=\{f\}$, and consequently we get $\Omega(f)=$ $=\mathcal{M}p(f)=\mathcal{M}(\{f\})=f$.

On the other hand let $\Omega(f)=f$. Then $\mathcal{M}(p(f))g=fg$ i.e. $\mathcal{M}(p(f)g)=fg$. Further, $\mathcal{M}(p(f))=fg$ then follows $f=fg$.

According to lemma 2. the equation $\Omega(f)=f$ is reproductive and according to theorem 1. in [2] every solution is of the form $\Omega(u)$, $u \in \mathrm{Mor}(A,B)$ that means we have next the *consequence*:

$$f \in \mathcal{Y}_g \Leftrightarrow (\exists u)\big(f=\Omega(u) \wedge u \in \mathrm{Mor}(A,B)\big) \quad \text{i.e.}$$

$$\mathcal{Y}_g=\{\Omega(u) \mid u \in \mathrm{Mor}(A,B)\}$$

Let us call the mapping Ω, which has the preceding properties, $\mathcal{M}_g$ — reproductive.

Example 2. Let R be the set of real numbers and $A=B=R$ and let $g^2=i_R$. Then $\mathcal{P}_g=\big\{\{h,hg\}/R \xrightarrow{h} R\big\}$. For $\tilde{h}=\{h,hg\}$ let us intrduce $\mathcal{M}(\tilde{h})=\dfrac{h+hg}{2}$. Obviously it is easy to check that $\mathcal{M}$ is of the form (3). Consequently $\Omega(h)=\dfrac{h+hg}{2}$. Then every solution of the equation (1) within category of:

(i) sets (then automorphisms are bijections),

(ii) topological spaces (in this case automorphisms are homeomorphisms),

(iii) the only object of category is R, and morphisms are polynomes (all automorphisms are polynomes in the form of $g(x)=a-x$ and $i_R(x)=x$) which have the form $\dfrac{h+hg}{2}$, where h is the morphism of the corresponding category.

Observe that in all three preceding cases the form of the solution is the same. Reason of that is in the fact that the functor between two categories transfers the equation (1).

III. In this section, we shall prove that if $\mathcal{Y}_g \neq \emptyset$ then exists $\mathcal{M}_g$ reproductive mapping. We can ask the question whether for equation (1) every idempotent mapping, which induces the corresponding reproductive equation, is $\mathcal{M}_g$ reproductive. We shall give answer to that question, too.

Let $\mathcal{Y}_g \neq \emptyset$. Thus from the theorem 2. [2] there exists the idempotent mapping $\Phi: \mathrm{Mor}(A,B) \to \mathrm{Mor}(A,B)$ such that the reproductive equation $\Phi(f)=f$ is equivalent to the equation (1), i.e.:

$$(\exists\, \Phi)\Big(\Phi\Phi=\Phi \wedge \big(f=fg \Leftrightarrow f=\Phi(f)\big)\Big)$$

Notice that $\Phi(f)g=\Phi(f)$ for all $f \in \mathrm{Mor}(A,B)$. Let S be a set of representants from every class $\tilde{f}$ (we use axiom of choice). As $\tilde{f}=\{f\}$ if and only if f is the solution of equation (1), S involves all solutions of equation (1). Let $p': \mathcal{P}_g \to S$, so that $p'(\tilde{f})=h$ where $h \in \tilde{f} \cap S$. Further, let us denote by j the embedding of the set S into $\mathrm{Mor}(A,B)$, i.e. $j(f)=f$ for $f \in S$.

Let us introduce the mapping $\mathcal{M}=\Phi jp'$ and $\Omega=\mathcal{M}p$. Then the following diagrams commutate:

$$\begin{array}{ccc} S & \xrightarrow{\;j\;} & \mathrm{Mor}(A,B) \\ {\scriptstyle p'}\uparrow & & \downarrow{\scriptstyle \Phi} \\ \mathcal{P}_g & \xrightarrow{\;\mathcal{M}\;} & \mathrm{Mor}(A,B) \end{array} \qquad \begin{array}{ccc} \mathcal{P}_g & \xleftarrow{\;\;p\;\;} & \mathrm{Mor}(A,B) \\ {\scriptstyle \mathcal{M}}\searrow & & \swarrow{\scriptstyle \Omega} \\ & \mathrm{Mor}(A,B) & \end{array}$$

Then we have $\mathcal{M}(\tilde{f})g=(\Phi jp'(\tilde{f}))g=\Phi(f')g=\Phi(f')=\mathcal{M}(\tilde{f})$, where $f'\in S\cap\tilde{f}$. As $\tilde{f}*g=\tilde{f}$, there is $\mathcal{M}(\tilde{f}*g)=\mathcal{M}(\tilde{f})$. Consequently we have:

(i) $\mathcal{M}(\tilde{f}*g)=\mathcal{M}(\tilde{f})g$ i.e. $\mathcal{M}$ is a G-mapping.

(ii) If $\tilde{f}=\{f\}$, then $f\in S$ and f satisfy the equation (1) and therefore $\Phi(f)=f$. That implies $\mathcal{M}(\{f\})=\Phi(f)=f$.

According to the preceding, $\mathcal{M}$ is of the form (3). Then using lemmas 2. and 3. we have that Ω is $\mathcal{M}_g$ reproductive. Then the following comes out:

Theorem 1. $\mathcal{Y}_g\neq\emptyset \Rightarrow (\exists\,\Omega)$ (Ω is $\mathcal{M}_g$ reproductive)

Lemma 4. (i) $\Omega|S=\Phi|S$

(ii) $(\forall f)\,(\exists f')\,((f,f')\in\rho\wedge\Omega(f)=\Phi(f'))$.

Proof. Really, if $f\in S$ then $\Omega(f)=\mathcal{M}(\tilde{f})=\Phi(f')$ where $f'\in S$ and $(f,f')\in\rho$. But, as $f\in S$ there is $f=f'$ that implies (i). Then (ii) is a direct consequence of (i).

Theorem 2. (i) Φ is $\mathcal{M}_g$ *reproductive if and only if* $\rho\subseteq\mathrm{Ker}\,\Phi$

(ii) *If* $\Phi=\mathcal{M}p$ *and* $\Phi=\mathcal{M}'p$ *then* $\mathcal{M}=\mathcal{M}'$.

Proof. Let Φ be $\mathcal{M}_g$ reproductive. If $(f,f')\in\rho$ then $\tilde{f}=\tilde{f}'$. Thus $\mathcal{M}(\tilde{f})=\mathcal{M}(\tilde{f}')$ i.e. $\Phi(f)=\Phi(f')$. Consequently $\rho\subseteq Ker\,\Phi$.

Inversely, let $\rho\subseteq\mathrm{Ker}\,\Phi$. Then exists the *unique* mapping $\tilde{\Phi}:\mathcal{P}_g\to\mathrm{Mor}(A,B)$ such that $\Phi=\tilde{\Phi}p$. If $\tilde{f}=\{f\}$ then f is a solution of the equation (1) and $\tilde{\Phi}(\tilde{f})=\Phi(f)=f$. Further, $\tilde{\Phi}(\tilde{f})g=\Phi(f)g=\Phi(f)=\tilde{\Phi}(\tilde{f})=\tilde{\Phi}(\tilde{f}*g)$ and $\tilde{\Phi}$ is of the form (3). That exactly means that Φ is $\mathcal{M}_g$ reproductive. That is the proof for (i). If $\Phi=\mathcal{M}p$, then, according to preceding, $\mathcal{M}=\tilde{\Phi}$, and (ii) is proved.

Let us show that in general case Φ may not be $\mathcal{M}_g$ reproductive. In that meaning, let us suppose that the equation (1) has at least two different solutions f and f' and there is morphism h such that $h\neq hg$. Let H map $\mathrm{Mor}(A,B)\setminus\mathcal{Y}_g$ in $\mathcal{Y}_g$, which satisfies $H(h)=f$ and $H(hg)=f'$. Now, we introduce the mapping Φ:

$$\Phi(u)=\begin{cases} u, & u\in\mathcal{Y}_g \\ H(u), & u\notin\mathcal{Y}_g \end{cases}$$

Then the equation $\Phi(f)=f$ is, according to theorem 2.[2], reproductive and equivalent to the equation (1). Further, $h*G=hg*G$ that implies $(h,hg)\in\rho$. On the other hand $\Phi(h)=f$ and $\Phi(hg)=f'$, i.e. $\Phi(h)\neq\Phi(hg)$. Hence, with regard to theorem 2. Φ can not be $\mathcal{M}_g$ – reproductive.

Example 3. Let $A=B=\{0,1\}$ and $g=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$. In category of sets $\operatorname{Mor}(A,B)=\{f_1, f_2, f_3, f_4\}$, where $f_1=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$, $f_2=\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}$, $f_3=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, $f_4=\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}$. Then $\tilde{f}_1=\{f_1\}$, $\tilde{f}_2=\tilde{f}_3=\{f_2, f_3\}$, $\tilde{f}_4=\{f_4\}$ and so $\mathcal{P}_g=\{\{f_1\}, \{f_4\}, \{f_2, f_3\}\}$.

On the other hand it is easy to find that $\mathcal{Y}_g=\{f_1, f_4\}$. Because of the property (i) and lemma 1. all $\mathcal{M}$ mappings are:

$$\mathcal{M}_1=\begin{pmatrix}\{f_1\} & \{f_4\} & \{f_2, f_3\}\\ f_1 & f_4 & f_1\end{pmatrix},\quad \mathcal{M}_2=\begin{pmatrix}\{f_1\} & \{f_4\} & \{f_2, f_3\}\\ f_1 & f_4 & f_4\end{pmatrix}.$$

Hence, all $\mathcal{M}_g$ mappings are:

$$\Omega_1=\begin{pmatrix}f_1 & f_2 & f_3 & f_4\\ f_1 & f_1 & f_1 & f_4\end{pmatrix},\quad \Omega_2=\begin{pmatrix}f_1 & f_2 & f_3 & f_4\\ f_1 & f_4 & f_4 & f_4\end{pmatrix}.$$

According to theorem 2.[2] the following idempotent mappings also solve the equation (1), but are not $\mathcal{M}_g$ – reproductive:

$$\Phi_1=\begin{pmatrix}f_1 & f_2 & f_3 & f_4\\ f_1 & f_1 & f_4 & f_4\end{pmatrix},\quad \Phi_2=\begin{pmatrix}f_1 & f_2 & f_3 & f_4\\ f_1 & f_4 & f_1 & f_4\end{pmatrix}.$$

BIBLIOGRAPHY

[1] S. Prešić, *Sur l'équation fonctionnelle,* $f(x)=f[g(x)]$, Publ. Elek. Fak. Univ. u Beogradu, 64, p. 29—31, 1961.

[2] S. B. Prešić, *Une classe d'équations matricielles et l'équation fonctionelle* $f^2=f$, Publ. Inst. Math., T. 8 (22), 1968, Beograd.

[3] S. Leng, *Algebra,* izd. „MIR“, Moskva, 1968.