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THE NUMBER OF ANTICHAINS OF FINITE POWER SETS

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Let $X=\{x_1, \ldots x_n\}$ be a finite set. The power set $P(X)$ of the set X is, as it is known, ordered by the inclusion. X_1 and X_2 $(X_1, X_2 \in P(X))$ are incomparable if neither $X_1 \subset X_2$ nor $X_2 \subset X_1$. The set $A=\{X_1, \ldots, X_k\}$ containing k mutually different elements of the set $P(X)$ is called antichain of length k of $(P(X), \subset)$ if every two sets X_i, X_j $(i \neq j)$ from A are incomparable.

Let $A(n, k)$ be the number of antichains of length k of the set $P(X)$, where $|X|=n$. Trivially we have $A(n, 1)=2^n$. The formulas for $A(n, 2)$ and $A(n, 3)$ are given in [1].

In this paper we shall present a procedure by which the explicite formula for $A(n, k)$ can be obtained for every given k (naturally depending on n). For some values of k the mentioned formula for $A(n, k)$ is deduced by use of a computer. The above is the contents of the first part of the paper. In the second part we represent another way for obtaining the results of [1].

In the paper the language and methods of the graph theory are used, because in this way the essence of the problem can be more easily seen.

1. Antichains and walks in a graph

The mapping $f: P(X) \rightarrow \{\beta_1, \ldots, \beta_n) \mid \beta_i \in \{0, 1\}, i=1, \ldots, n\}$ defined by

$$(\forall\, Y \in P(X))\, \big(fY=(\beta_1, \ldots, \beta_n)$$

$$\Leftrightarrow (\forall_i \in \{1, \ldots, n\})\, (x_i \in Y \Rightarrow \beta_i = 1 \wedge x_i \notin Y \Rightarrow \beta_i = 0)\big)$$

is a bijection. If a walk of length k in a graph without multiple edges is identified with n-tuple of vertices through which it passes, each of the above n-tuples determines one walk of length $n-1$ in the graph of Fig. 1.

The number of such walks is obviously equal to 2^n and this is the number of all n-tuples $(\beta_1, \ldots, \beta_n)$ or the number of elements of $P(X)$ or, finally, the number $A(n, 1)$ of antichains of length 1 of $P(X)$.

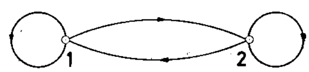

Fig. 1

We shall connect, for arbitrary k, the number $A(n, k)$ with the number of walks of length $n-1$ having certain properties in a suitably chosen graph. An antichain of length k of $P(X)$ can be determined by the matrix $\mathcal{A}=\| a_{ij} \|_{k, n}$ $(a_{ij} \in \{0, 1\})$. Every row of the matrix $\mathcal{A}$ (interpreted as an n-tuple) determines, in the above mentioned sense, one element of the antichain $A=\{X_1, \ldots, X_k\}$. To every antichain of length k corresponds $k!$ such matrices, while, naturally, there are matrices not determining an antichain.

The matrix $\mathcal{A}$ can be interpreted as the n-tuple of its columns. Therefore, the number of all matrices $\mathcal{A}$ is equal to the number of walks of length $n-1$ in the graph containing all possible edges and loops and whose vertices are all possible columns of matrices $\mathcal{A}$.

The columns are, in fact, k-tuples of symbols 0 and 1. Since the number of all possible columns is equal to 2^k, the corresponding graph has 2^k vertices. For $k=2$ it is represented on Fig. 2. In this case the number of walks of length $n-1$, i. e., the number of matrices $\mathcal{A}$ is equal to 4^n. In the further text we shall encounter always with the graphs containing all possible edges and loops. If such a graph has l vertices, then it has, obviously, l^n walks of length $n-1$.

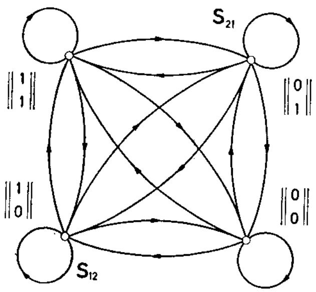

Fig. 2

For $k=2$ we shall determine the number of antichains by means of a method which will be then applied to the general case. In the graph on Fig 2 we shall determine the number of walks defining the matrices $\mathcal{A}$ corresponding to antichains of $P(X)$. It can be easily seen that the matrix $\mathcal{A}$ (of the type $2 \times n$) determines the antichain of length 2 if and only if it contains at least one column of the form $\begin{Vmatrix}1\\0\end{Vmatrix}$ and at least one of the form $\begin{Vmatrix}0\\1\end{Vmatrix}$. Therefore, the walk in the graph on Fig. 2 determines an antichain if and only if it passes at least once through the vertex $\begin{Vmatrix}1\\0\end{Vmatrix}$ and at least once through the vertex $\begin{Vmatrix}0\\1\end{Vmatrix}$.

Let N_t denote the number of all walks of length $n-1$ in the graph. If S is a subset of the set of vertices of a graph, N_s denotes the number of walks (of length $n-1$) passing at least once through at least one vertex from S. For such walks we say that they pass through the set S or that they have the property s. In general, if a subset of the set of vertices is denoted by a capital letter, the corresponding property will be denoted by a small letter. $\bar{s}$ denotes the property opposed to s, i. e., $N_{\bar{s}}$ denotes the number of walks not having the property s. Hence, $N_t = N_s + N_{\bar{s}}$. If s_1 and s_2 are two properties of the walks, $N_{s_1 s_2}$ denotes the number of walks that have the property s_1 and the property s_2 also.

If in the graph on Fig. 2 we take $S_{12}=\left\{\begin{Vmatrix}1\\0\end{Vmatrix}\right\}$ and $S_{21}=\left\{\begin{Vmatrix}0\\1\end{Vmatrix}\right\}$ we get

$$2!\, A(n, 2) = N_{s_{12} s_{21}}.$$

On the basis of the combinatorial inclusion-exclusion principle we can write

$$N_{s_{12} s_{21}} = N_t - N_{\bar{s}_{12}} - N_{\bar{s}_{21}} + N_{\bar{s}_{12} \bar{s}_{21}}.$$

It can be easily seen that $N_{\bar{s}_{12}}$ is equal to the number of all walks in the graph obtained by deleting vertices of the set S_{12} from the initial graph. In the same way it holds for $N_{\bar{s}_{21}}$, while $N_{\bar{s}_{12}\bar{s}_{21}}$ is equal to the number of all walks in the graph obtained by deleting the vertices of the set $S_{12} \cup S_{21}$ from the initial graph. By deleting vertices from the complete graph we again get a complete graph and the determining the number of walks is simple, as it has been already mentioned. Since $N_t = 4^n$, $N_{\bar{s}_{12}} = N_{\bar{s}_{21}} = 3^n$ and $N_{\bar{s}_{12}\bar{s}_{21}} = 2^n$, we get

$$A(n, 2) = \frac{1}{2} 4^n - 3^n + \frac{1}{2} 2^n.$$

The above quoted idea can be simply applied also in the case of the antichains of the arbitrary length k. We consider at first the complete graph with 2^k vertices representing the ordered k-tuples of elements 0 and 1, i. e., all possible columns of matrices $\mathcal{A}$. For every ordered pair of indices i, j ($i \neq j$, $i, j = 1, \ldots, k$) we define the set S_{ij} as the set of all k-tuples (columns) whose element on the i-th place is equal to 1 and whose element on the j-th place is equal to 0. The passing of a walk through the sets S_{ij} and S_{ji} ensures that the elements of the set $P(X)$ determined by i-th and j-th row of the corresponding matrix $\mathcal{A}$ are incomparable. It can be easily seen that

$$k!\, A(n, k) = N_{s_{12} s_{13} \ldots s_{ij} \ldots s_{k1}},$$

where the index pair ij passes through all the 2-permutations (without repetitions) of the set $\{1, \ldots, k\}$.

The principle of inclusion and exclusion yields now the following

Theorem. *We have*

$$A(n, k) = \frac{1}{k!} \left(N_t - \Sigma N_{\bar{s}_{ij}} + \Sigma N_{\bar{s}_{ij}\bar{s}_{lm}} - \cdots + (-1)^k N_{\bar{s}_{12} \ldots \bar{s}_{k1}}\right), \tag{1}$$

where: $\Sigma N_{\bar{s}_{ij}}$ *denotes the sum over all properties* $\bar{s}_{ij}$, $\Sigma N_{\bar{s}_{ij}\bar{s}_{lm}}$ *denotes the sum over all combinations of the second class of properties* $\bar{s}_{ij}$, *etc.*

On the basis of (1) a computer programme for determining the explicite formula for $A(n, k)$ (for the given k) was made. The programme generates the sets S_{ij} and all their combinations. For each combination the union of sets forming the combination is determined. If the combination is of the m-th class and the mentioned union contains l elements, the contribution of the considered combination to the number $k!\,A(n, k)$ is equal to $(-1)^m (2^k - l)^n$. The programme has been realized by N. Klem.

The programme provides the result for $k = 3$ instantaneously. The result of M. Popadić was obtained and this was, among other things, a test for the correctness of the programme. For $k = 4$ an IBM 1130 computer provided the result for about 10 minutes. For greater values of k the time for computer solution would be much longer if the algorithm were not improved.

We list now the known formulas for $A(n, k)$ as well as the formula obtained for $k = 4$:

$$A(n, 1) = 2^n$$

$$A(n, 2) = \frac{1}{2} 4^n - 3^n + \frac{1}{2} 2^n,$$

$$A(n,\ 3)=\frac{1}{6}\,8^n-6^n+5^n+\frac{1}{2}\,4^n-3^n+\frac{1}{3}\,2^n,$$

$$A(n,\ 4)=\frac{1}{24}\,16^n-\frac{1}{2}\,12^n+10^n+\frac{1}{6}\,9^n-\frac{3}{4}\,8^n+\frac{1}{4}\,7^n-\frac{3}{2}\,6^n$$

$$+\frac{3}{2}\,5^n+\frac{11}{24}\,4^n-\frac{11}{12}\,3^n+\frac{1}{4}\,2^n.$$

2. Another way for determining the number of antichains

It is of interest to note that M. Popadić's result can be deduced by means of graph theory in another way, too.

For indirected graphs without loops or multiple edges the following formula (which was otherwise in a similar form noticed in [2]) obviously holds

$$E+F+\frac{1}{2}\sum_{j=1}^{L}(N-1-j)\,jp_j=\binom{N}{3}. \tag{2}$$

Here N represents the number of vertices of the graph, L the number of edges, p_j the number of vertices of degree j, E the number of induced subgraph having three vertices and no edges and F the number of triangles.

Consider the graph G whose vertices are in $1-1$ correspondence with the elements of the set $P(X)$. Two vertices are adjacent if and only if the corresponding elements are comparable in $(P(X),\ \subset)$. The number of antichains of length 3 is then equal to the quantity E in (2). It can be easily seen that $N=2^n$ and

$$\frac{1}{2}\sum_{j=1}^{L}(N-1-j)\,jp_j=\frac{1}{2}\sum_{i=0}^{n}(2^n-2^{n-i}-2^i+1)(2^{n-i}+2^i-2)\binom{n}{i}$$

$$=6^n-5^n-2\cdot 4^n+3\cdot 3^n-2^n.$$

F is still to be determined. Consider the sets $A,\ B,\ C$ $(A,\ B,\ C\in P(X))$ corresponding to vertices of a triangle from G. Then holds $A\subset B\subset C$ or a similar relation in which the sets $A,\ B,\ C$ appear in a different order. Hence, the number of triangles is equal to the number of chains of length 3. The set $B\in P(X)$ for which $|B|=i$ has exactly 2^i-1 proper subsets and exactly $2^{n-i}-1$ proper oversets.

Therefore, B is the mean element in exactly $(2^i-1)(2^{n-i}-1)$ chains of length 3. The number of all chains of length 3 is then equal to $\sum_{i=0}^{n}\binom{n}{i}(2^i-1)(2^{n-i}-1)=4^n-2\cdot 3^n+2^n$.

Putting the obtained expressions in (2) we get the result [1] for $A(n,\ 3)$.

Notice that in the case of bipartite graphs we have $F=0$ and the formula (2) expresses the number of induced subgraphs with three vertices and no edges, in fact, in terms of vertex degrees.

As an example of application of (2) to biparite graphs we determine the number of all arrangements of 3 knights (chess figures) on a chess board of dimensions $n\cdot n$ in which they do not attack each other. As usual to each chess figure we correspond a graph. The vertices of the graph are in $1-1$

correspondence with the fields of chessboard. Two vertices are joined by an edge if and only if the figure can make a move between the corresponding fields of the chess board. The graph corresponding to the knight is bipartite, because the knight always goes from the white to the black field and vice versa.

Consider one corner of the chessboard and write in every field the number of moves which knight can make if it is standing in that field. Then we get the following schema:

$$\begin{matrix} 2 & 3 & 4 & 4 & \cdot & \cdot & \cdot \\ 3 & 4 & 6 & 6 \\ 4 & 6 & 8 & 8 \\ 4 & 6 & 8 & 8 \\ \cdot \\ \cdot \\ \cdot \end{matrix}$$

The corresponding graph obviously has (for $n \geqslant 4$) 4 vertices of degree 2, 8 vertices of degree 3, $4(n-4)+4=4n-12$ vertices of degree 4, $4(n-4)$ vertices of degree 6 and $(n-4)^2$ vertices of degree 8. On the basis of (2) we have for the number E of the arrangement of three knights on an $n \cdot n$ chessboard in which they do not attack each other, the following expression

$$E=\binom{n^2}{3}-\frac{1}{2}[(n^2-3)\cdot 2\cdot 4+(n^2-4)\cdot 3\cdot 8+(n^2-5)\cdot 4\cdot(4n-12)+(n^2-7)\cdot 6 \cdot(4n-16)+(n^2-9)\cdot 8\cdot(n-4)^2],$$

i. e.

$$E=\frac{1}{6}(n-2)(n^5+2n^4-23n^3+26n^2+222n-540). \tag{3}$$

This result was otherwise noted in the chess problem literature (see [3], p. 61, where the formula (3) is given without proof).

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