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FUNCTIONS WITH ASYMPTOTICALLY INFINITE DIFFERENCES

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A well-known result, due to Karamata ([1], [2]), is that if f is a continuous function on the reals and if $f(x+t)-f(x)$ converges as $x \to +\infty$ for every t real, then the convergence is uniform in t on bounded sets. Strengthened, generalized, and analogous results have been proved. (see, e. g., [3], [4], [5].) In this paper we obtain one more analogous result; namely, if f is a measurable, real-valued function on the reals and if $f(x+t)-f(x)$ diverges to $+\infty$ as $x \to +\infty$ for every t real and positive, then this divergence is uniform in t on $[\delta, \infty]$ for every δ positive.To state and prove this result in a more general form, we introduce the following notation:

$$(R^n)^+ = \{(r_1, r_2, \ldots, r_n) : r_i > 0 \text{ for } i = 1, \ldots, n\},$$

m denotes Lebesgue measure on $(R^n)^+$,

for t real: $\quad T_t f(x) = f(x+t),\ \Delta_t f(x) = f(x+t) - f(x),$

for integers $\quad j > 1:\ \Delta_t^j f(x) = \Delta_t^{j-1}(\Delta_t f(x))$

for $\quad \sigma = (s_1, s_2, \ldots, s_n)$ in $(R^n)^+ : \Delta_\sigma f(x) = \Delta_{s_1} \Delta_{s_2} \cdots \Delta_{s_n} f(x),$

for a subset A of $(R)^+ : A^n$ denotes the n-fold cartesian probuct of A with itself.

Theorem. *If f is a real-valued, measurable function on R^+ which satisfies*

(1) $$\Delta_\sigma f(x) \to +\infty \text{ for all } \sigma \text{ in } (R^n)^+,$$

then

(i) *the divergence in* (1) *is uniform in σ on subsets S of $(R^n)^+$ which are bounded away from the axes,*

(ii) *for some X positive, f is bounded on every finite sub-interval of $[X, \infty)$,*

(iii) *$f(x) \to +\infty$ and $\Delta_\sigma f(x) \to +\infty$ as $x \to +\infty$ for all*

$$\sigma \text{ in } (R^j)^+,\ j = 1, \ldots, n.$$

Proof of (i). We need the following identities:

(2) $$\Delta_\sigma = \sum_{d_1=0}^{1} \cdots \sum_{d_n=0}^{1} \left(\prod_{j=1}^{n} (T_{b_j} \Delta_{s_j - b_j})^{d_j} (\Delta_{b_j})^{1-d_j} \right),$$

where $b_1, \ldots, b_n$ are arbitrary real numbers,

$$(3) \qquad \Delta_\lambda = \sum_{k_n=1}^{j_n} \cdots \sum_{k_1=1}^{j_1} \Delta_\sigma T_{q(k_1, \ldots, k_n;\, \sigma)},$$

where $\lambda = (j_1 s_1, \ldots, j_n s_n)$; $j_1, \ldots j_n$ are positive integers; and $q(k_1, \ldots, k_n; \sigma) = \sum_{i=1}^{n} (k_i - 1) s_i$,

$$(4) \qquad (T_z \Delta_{s-b})^d (\Delta_b)^{1-d} = (T_{s-b})^d \text{ where } z = \begin{cases} s-b, & \text{if } d=1 \\ b \quad , & \text{if } d=0 \end{cases}$$

We obtain (2) by taking the product of the n identities

$$\Delta_{s_j} = T_{b_j} \Delta_{s_j - b_j} + \Delta_{b_j}, \qquad j = 1, 2, \ldots, n.$$

We obtain (3) by taking the product of the n identities

$$\Delta_{j_i s_i} = \sum_{j=1}^{j_i} \Delta_{s_i} T_{(j-1)s_i}, \qquad j = i, \ldots, n,$$

and observing that the product of the right-hand sides is equal to

$$\sum_{k_1=1}^{j_n} \cdots \sum_{k_1=1}^{j_1} \prod_{i=1}^{n} T_{(k_i-1)s_i}$$

and then noting that $\prod_{i=1}^{n} \Delta_{s_i} T_{(k_i-1)s_i}$ equals $\Delta_\sigma T_{q(k_1, \ldots, k_n;\, \sigma)}$.

The identity (4) is easily verified in each of the cases $d=0$ and $d=1$.

To prove (i) it suffices to consider the case $S=[a, 2a]^n$, where a is a positive number. To see that this case implies the general case we have only to show that if $\Delta_\sigma f(x) > m$ for all σ in $[a, 2a]^n$ and $x > X$, then $\Delta_\lambda f(x) > m$ for all λ in $[a, \infty)^n$ and $x > X$. For this we observe that there exist $\sigma = (s_1, \ldots, s_n)$ in $[a, 2a]^n$ and positive integers $j_1, \ldots, j_n$ such that $\lambda = (j_1 s_1 \ldots, j_n s_n)$; that, by (3),

$$\Delta_\lambda f(x) = \sum_{k_n=1}^{j_n} \cdots \sum_{k_n=1}^{j_1} \Delta_\sigma f\left(x + \sum_{i=1}^{n} (k_i - 1) s_i\right);$$

and that each of the terms in the last sum is greater than m for x greater than X.

We now consider the set $S=[a, 2a]^n$, where a is some positive number, and assume that the divergence in (1) is not uniform in S. This implies that for $r=1, 2, \ldots$, there exist $\sigma(r) = (s_1(r), \ldots, s_n(r))$ in $[a, 2a]^n$ and real x_r, and a constant K such that $x_r \to +\infty$ and

$$(5) \qquad \Delta_{\sigma(r)} f(x_r) < K \text{ for } r = 1, 2, \ldots.$$

Let $B = [a/4, a/2]^n$. For $\delta = (d_1, \ldots, d_n)$ in $\{0, 1\}^n$, define $B_{\delta, r}$ to be the set of $(b_1, \ldots, b_n)$ in B satisfying

$$\left(\prod_{j=1}^{n} (T_{b_j} \Delta_{s_j(r) - b_j})^{d_j} (\Delta_{b_j})^{1-d_j}\right) (f(x_r)) < K/2^n$$

From (2) and (5) we obtain $B = \bigcup_{d_n=0}^{1} \cdots \bigcup_{d_1=0}^{1} B_{\delta,r}$ for $r = 1, 2, \ldots$. So for some δ in $\{0, 1\}^n$

(6) $$m(B_{\delta,r}) \geqslant (mB)/2^n = (a/8)^n \text{ for infinitely many } r.$$

Hereafter we consider a fixed δ satisfying (6). Define Z_r to be the set of $(z_1, \ldots, z_n)$ in $(R^n)^+$ satisfying

$$z_j = \begin{cases} s_j(r) - b_j(r), & \text{if } d_j = 1 \\ b_j(r) & \text{, if } d_j = 0 \end{cases}$$

for some $(b_1(r), \ldots, b_n(r))$ in $B_{\delta,r}$ and $j = 1, \ldots, n$. Observe that Z_r is obtained from $B_{\delta,r}$ by reflections and translations so that Z_r is measurable with the same measure as $B_{\delta,r}$. Thus, by (6),

(7) $$mZ_r \geqslant (a/8)^n \text{ for infinitely many } r.$$

Since $B_{\delta,r} \subseteq B$ for $r = 1, 2, \ldots$, where B is a fixed bounded set, and since for $r = 1, 2, \ldots$ the translations in the definition of Z_r are all by elements from the fixed bounded set S, we obtain that $\bigcup_{r=1}^{\infty} Z_r$ has finite measure. This and (7) imply

$$m(\limsup Z_r) = m\Big(\lim_{k\to+\infty} \bigcup_{r>k} Z_r\Big) = \lim_{k\to\infty} m\Big(\bigcup_{r>k} Z_r\Big) \geqslant (a/8)^n,$$

so that there exists $z = (z_1, \ldots, z_n)$ in $(R^n)^+$ such that z is an element of Z_r for infinitely many r. Equivalently, there is a z in $(R^n)^+$ such that for infinitely many r there exist $(b_1(r), \ldots, b_n(r))$ in $B_{\delta,r}$ having the property

$$z_j = \begin{cases} s_j(r) - b_j(r), & d_j = 1 \\ b_j(r) & , d_j = 0 \end{cases} \quad \text{for } j = 1, \ldots, n.$$

From this, by applying (4) to every term of the product

$$\prod_{j=1}^{n} (T_{b_j(r)} \Delta_{s_j(r)-b_j(r)})^{d_j} (\Delta_{b_j(r)})^{1-d_j},$$

and by recalling the definition of $B_{\delta,r}$ we obtain that

$$\prod_{j=1}^{n} ((T_{s_j(r)-z_j})^{d_j} \Delta_{z_j}) ((f(x_r)) < K/2^n$$

for infinitely many r. This last inequality can be written in the form $\Delta_z f(w_r) < K/2^n$ where $z = (z_1, \ldots, z_n)$ is in $(R^n)^+$ and $w_r = x_r + \sum_{j=1}^{n} d_j (s_j(r) - z_j) \to +\infty$ as $r \to +\infty$. This contradicts (1), proving (i).

Proof of (ii). Let $0 < a < b$ be arbitrary. We use (i) to choose $X > a$ so that

(8) $$\Delta_t f(x) > 0 \text{ for } x > X, \; t \text{ in } [a, b].$$

Suppose that f is unbounded on some sub-interval $I = [c, d]$ of $[X, \infty)$. Without loss of generality we assume that I has length $b - a$. There exists a sequence x_r of elements from I such that either

(9) $$f(x_r) > r, \text{ for } r = 1, 2, \ldots, \text{ or}$$

(10) $$f(x_r) < -r, \text{ for } r = 1, 2, \ldots.$$

If (9) holds, choose t_r from $[a, b]$ so that $x_r+t_r=d+a$, $r=1, 2, \ldots$. Then (8) and (9) imply that

$$f(d+a)=f(x_r+t_r)>f(x_r)>r, \quad r=1, 2, \ldots,$$

contradicting $f(d+a)<+\infty$. If (10) holds, choose t_r from $[a, b]$ so that $x_r-t_r=c-a$. Then (8) and (10) imply

$$f(c-a)=f(x_r-t_r)<f(x_r)<-r, \quad r=1, 2, \ldots,$$

contradicting $f(c-a)>-\infty$.

Proof of (iii). We show that $\Delta_t f(x)\to+\infty$, for all t in R^+, implies $f(x)\to+\infty$ as $x\to+\infty$. By repeated applications of this result we shall obtain (iii).

By (i) and (ii) we can choose $X>1$ so that

(11) $$\Delta_t f(x)>1, \text{ for } x>X, \ t \text{ in } [1, \infty), \text{ and}$$

(12) $$f(x)>C, \text{ for } x \text{ in } [X, 2X] \text{ where } C \text{ is a fixed constant.}$$

Let k be a positive integer, and x_0 an element of (kX, ∞). Then $x_0=px_1$ for some integer $p>k$ and x_1 in $[X, 2X]$, so

(13) $$f(x_0)=f(px_1)=f(x_1)+\sum_{j=1}^{p-1}\Delta_{x_1}f(jx_1).$$

By (11), each term in this last sum is greater than one. Since $p>k$, (12) and (13) imply $f(x_0)>k+C$ for arbitrary x_0 in $[kX, \infty)$. This completes the proof (iii) and the theorem.

Remarks.

(A) The conclusion of the theorem does not necessarily hold if we remove the requirement that S be bounded away from the axes. To see this, let $f(x)=x^2$, $S=\{1/r\colon r=1, 2, \ldots\}$, $t_r=1/r$, $x_r=r$. Then $\Delta_t f(x)\to+\infty$ as $x\to+\infty$ for t in R^+, but $\Delta_{t_r}f(x_r)\to 2$ as $r\to+\infty$.

(B) The proof of (ii) of the theorem actually yields more than is stated; namely, if f is a real-valued not necessarily measurable function on R^+, and if there is no X such that f is locally bounded on $[X, \infty)$, and if $\Delta_t f(x)\to+\infty$ as $x\to+\infty$ for every t in R^+, then on every interval of R^+ the divergence of $\Delta_t f(x)$ as $x\to+\infty$ is not uniform in t.

(C) The conclusion of the theorem does not necessarily hold if we remove the requirement that f be measurable, even if we require that S be a compact subset of R^+. To see this, let $g(x)$ be any non-measurable solution of Cauchy's equation, and let $f(x)=g(x)+x^2$. Then $\Delta_t f(x)\to+\infty$ as $x\to+\infty$ for all t in R^+, and f is unbounded on every sub-interval of R^+, so by remark (ii) the divergence is not uniform in t on any sub-interval of R^+.

(D) There remains one result of Baishanski [5] which we have been unable to extend. He proved that if $\Delta_t^n f(x)$ converges as $x\to+\infty$ for all t real, then $\Delta_\sigma f(x)$ converges as $x\to+\infty$ for all σ in R^n. Is it true that $\Delta_t^n f(x)\to+\infty$ as $x\to+\infty$ for all t in R^+ implies $\Delta_\sigma f(x)\to+\infty$ as $x\to+\infty$ for all σ in $(R^n)^+$?

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