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A PROOF OF A CONJECTURE BY KARAMATA

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A formulation of a conjecture by Karamata [1, p. 58] is that for any real-valued continuous function f the limit

(1) $$\lim_{x \to \infty} (f(x+t) - f(x))$$

exists for every t real if

(2) the limit in (1) exists for two positive reals with irrational quotient

and if there exists a function w defined on $(0, \infty)$ such that

(3) $$w(t) \to 0 \quad \text{as} \quad t \to 0 \quad \text{and}$$
$$\liminf (f(x+t) - f(x)) \geq w(t) \text{ for all } t \text{ in } (0, \infty).$$

In this paper we prove the strengthened version of Karamta's conjecture obtained by assuming f to be measurable rather than continuous and by replacing (2) with

(2′) the limit in (1) exists for all t in a dense subset D of the reals.

Condition (2′) is weaker than (2) since the set of t for which the limit in (1) exists is an additive subgroup of the reals.

For any real-valued function f defined on the reals and reals x and t define

$$\Delta_t f(x) = f(x+t) - f(x).$$

If t is such that the limit in (1) exists, then let $g_f(t)$ denote this limit. For any measurable subset A of the reals let $t - A$ be the set $\{t - a : a \in A\}$ and mA the Lebesgue measure of A.

The strengthened version of Karamata's conjecture follows immediately from the following theorem.

Theorem. *Let f be a real-valued, measurable function defined on the reals such that $g_f(t)$ is defined for every t in a dense subset D of the reals. Then the following are equivalent:*

(i) *$g_f(t)$ is defined for every t real,*

(ii) *given ε positive, there exist X and a positive so that t in (o, a) and $x > X$ imply $|\Delta_t f(x)| < \varepsilon$,*

(iii) *there exists a function w defined on $(0, \infty)$ satisfying (3),*

(iv) *there exists a real-valued function* u *defined on* $(0, \infty)$ *such that* $u(t) \to 0$ *as* $t \to 0$, *and*

$$\limsup_{x\to\infty} \Delta_t f(x) \leq u(t) \text{ for all } t \text{ in } (0, \infty).$$

Proof. The implication (i) $\Rightarrow$ (iv) is an immediate consequence of a result due to Karamata [1, p. 56] which after an easy extension from continuous to measurable functions can be formulated as follows: if f is a real-valued measurable function on the reals and $g_f(t)$ is defined for all real t, then $g_f(t) = At$ for some real constant A. Letting $u(t) = At$ we obtain (iv), and in a similar manner obtain (i) $\Rightarrow$ (iii). Thus, to complete the proof of the theorem it suffices to prove that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) since (iv) implies that (iii) is valid with f replaced by $-f$ and since $g_{-f}(t)$ being defined for all real t implies (i).

To prove the implication (ii) $\Rightarrow$ (i) let t_* and ε be given positive numbers. By (ii) there exist X and a positive so that

$$|\Delta_t f(x)| < \varepsilon \text{ for all } t \text{ in } (0, a) \text{ and } x > X$$

and by hypothesis there exist t_0 and Y positive so that

$$0 < t_* - t_0 < a,$$

and

$$|\Delta_{t_0} f(x) - \Delta_{t_0} f(y)| < \varepsilon \text{ for all } x, y > Y.$$

Thus, for $x, y > \max(X, Y)$

$$|\Delta_{t_*} f(x) - \Delta_{t_*} f(y)| \leq 3\varepsilon$$

since by the triangle inequality the left hand side of this last inequality is less than or equal to

$$|\Delta_{t_*-t_0} f(x+t_0)| + |\Delta_{t_0} f(x) - \Delta_{t_0} f(y)| + |\Delta_{t_*-t_0} f(y+t_0)|.$$

This proves that $g_f(t_*)$ is defined, hence the implication (ii) $\Rightarrow$ (i).

To complete the proof of the theorem we shall use that (iii) implies the following:

$$\text{(4)} \quad \begin{cases} \text{given } \varepsilon \text{ positive, there exists a positive constant } a \text{ such that} \\ \qquad\qquad g_f(d) < \varepsilon \\ \text{for all } d \text{ in } (0, a) \cap D. \end{cases}$$

To prove (4), use (iii) and that D is dense to choose a positive element d_0 of D such that

$$|w(x)| < \varepsilon/3 \text{ for all } x \text{ in } (0, d_0).$$

Let K be an arbitrary positive integer and d be in $D \cap (0, d_0/k)$. The last inequality and (iii) imply that there exists a positive constant X such that

$$\Delta_{d_0-kd} f(x+kd) \geqq -\frac{2\varepsilon}{3} \text{ for all } x > X.$$

Using the identity

$$\Delta_{d_0-kd} f(x+kd) = \Delta_{d_0} f(x) - \sum_{j=0}^{k-1} \Delta_d f(x+jd)$$

and letting x tend to infinity we obtain from the last inequality that

$$g_f(d) \leq \frac{g_f(d_0)}{k} + \frac{2\varepsilon}{3},$$

which for k sufficiently large proves (4).

Assume that (iii) and the negation of (ii) hold. Then there exist sequences $\{y_i\}$, $\{t_i\}$ of positive reals and a positive constant ε such that $y_i \to \infty$, $t_i \to 0$ and either

(5) $$\Delta_{t_i} f(y_i) \geq \varepsilon, \quad i = 1, 2, \ldots$$

or

(6) $$\Delta_{t_i} f(y_i) \leq -\varepsilon, \quad i = 1, 2, \ldots .$$

Use (iii) and that D is dense to choose d_0, a positive element of D, such that

(7) $$w(x) > -\varepsilon/4 \text{ for all } x \text{ in } (0, d_0),$$

where by (4) we can assume that

$$\Delta_{d_0} f(x) < \varepsilon/2$$

for all x greater than some positive constant X.

If (5) holds, then let

$$x_i = y_i + t_i, \quad r_i = d_0 - t_i,$$

otherwise let

$$x_i = y_i - d_0, \quad r_i = d_0 + t_i.$$

The last inequality, (5) and (6) imply that for all i sufficiently large

(8) $$\Delta_{r_i} f(x_i) \leq -\varepsilon/2$$

where $r_i \to d_0$ and $x_i \to \infty$.

Define

$$S_i = \left\{ t \in \left[\frac{d_0}{3}, \frac{2d_0}{3} \right] : \Delta_t f(x_i) \leq -\varepsilon/4 \right\},$$

$$T_i = \left\{ t \in \left[\frac{d_0}{3}, \frac{2d_0}{3} \right] : \Delta_{r_i - t} f(x_i + t) \leq -\varepsilon/4 \right\}$$

and obtain by (8) that for all i sufficiently large

$$[d_0/3, 2d_0/3] = S_i \cup T_i$$

so that either

(9) $$mS_i \geq \frac{d_0}{6} \text{ for infinitely many } i,$$

or

$$mT_i \geq \frac{d_0}{6} \text{ for infinitely many } i.$$

In the latter case note that there exists a point r_0 which is in $r_i - T_i$ for infinitely many i since $m(r_i - T_i) = mT_i$ and since $\bigcup_i (r_i - T_i)$ is contained in a set of finite measure. Thus, by the definition of the T_i's

$$\Delta_{r_0} f(x_i + r_i - r_0) \leq -\varepsilon/4$$

for infinitely many i, where $x_i+r_i-r_0\to\infty$ as $i\to\infty$ and where r_0 is in $\left[\frac{d_0}{3}, \frac{2d_0}{3}\right]$. This contradicts (7). By replacing r_i-T_i with S_i in the above argument we also obtain a contradiction if (9) holds.

REFERENCES

[1] J. Karamata, *Sur un mode de croissance régulière. Théorèmes fondamentaux,* Bull. de la Soc. Math. de France 61, (1933), 55—62.

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