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MATRIX METHODS WHICH SUM SEQUENCES OF BOUNDED k-VARIATION

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If k is a positive integer, let BV_k^* denote the set of all convergent complex sequences $z=\{z_p\}$ such that

$$(^\circ) \qquad \sum_{p=1}^{\infty}\binom{p+k-2}{k-1}|\Delta^k z_p|<\infty,$$

where $\Delta z_p=z_p-z_{p+1}$, $\Delta^2 z_p=\Delta z_p-\Delta z_{p+1}$, etc. We note that BV_1^* is the class (usually denoted by BV) of absolutely convergent complex sequences (or sequences of bounded variation). We shall say that a convergent sequence z for which $(^\circ)$ holds is of bounded k-variation. Actually BV_k^* is the set of elements of the complex linear space generated by the set of all convergent real sequences having nonnegative k-th differences [4]. It is easy to show that BV_j^* properly contains BV_{j+1}^*, $j=1, 2, 3, \ldots$ [4].

In this paper we give a constructive proof of the fact that if a complex matrix A sums every null sequence of bounded k-variation, then A sums „many" other null sequences. For $k=1$, this result follows from Theorem 1 of [3], and for $k=2$, the result is Theorem 3 of [3].

In [1] necessary and sufficient conditions were given for a complex matrix to sum every sequence in BV_k^*. Since we have no interest here in constant term sequences (which obviously belong to BV_k^* for all k), the main background for the present paper is contained in the following lemma, the proof of which is a trivial modification of the proof of the theorem in [1] and will be omitted.

Lemma 1. *A complex matrix $A=(a_{pq})$ sums every null sequence in BV_k^* if and only if the following conditions hold:*

(1) *A has convergent columns,*

(2) *there exists L_p such that* $\left|\sum_{q=1}^{n} a_{pq}\right|<L_p$, $n, p=1, 2, 3, \ldots$,

(3) *there exists L such that*

$$\left|\sum_{s_{k-1}=1}^{s_k=n}\sum_{s_{k-2}=1}^{s_{k-1}}\cdots\sum_{s_1=1}^{s_2}\sum_{q=1}^{s_1}a_{pq}\right|<\binom{n+k-2}{k-1}L, \quad n, p=1, 2, 3, \ldots .$$

The following notation was used in [3]. If $A=(a_{pq})$, $c=\{c_p\}$, and $d=\{d_p\}$, then $cd=\{c_p d_p\}$, and $A_c=B=(b_{pq})$, where $b_{pq}=a_{pq}c_q$. Also in [3], $\sum^{-1}A$, $\sum^{1}A$ and $\sum^{2}A$ were defined. We extend this notation by defining $\sum^{n}A=\sum^{1}(\sum^{n-1}A)$, $n>1$. We will use $\Delta^n x$ to denote the sequence $\{\Delta^n x_p\}_{p=1}^{\infty}$.

The following lemma is an obvious extension of Lemma 6 of [3].

Lemma 2. *Suppose* $A=(a_{pq})$ *is a matrix such that* $\left|\sum_{q=1}^{n} a_{pq}\right|<L_p$, $n, p=1, 2, 3, \ldots$, *and* x *is a null sequence. Then*

1) *if* Ax *or* $(\sum^{1}A)(\Delta x)$ *is defined, then* $Ax=(\sum^{1}A)(\Delta x)$,
2) *if* $m>1$ *and* Ax *or* $(\sum^{t}A)(\Delta^{t}x)$ *is defined for some* t, $1\leq t\leq m$, *and* $n^{s}\Delta^{s}x_n\to 0$ *as* $n\to\infty$, $s=1, 2, \ldots, m-1$, *then* $Ax=(\sum^{r}A)(\Delta^{r}x)$, $r=1, 2, \ldots, m$.

Theorem 1. *If* A *sums every null sequence in* BK_k^*, *then there exists a null sequence not in* BV_k^* *which* A *sums.*

Proof. Since the theorem is known for $k=1, 2$, assume $k>2$. Let $c=\left\{\binom{n+k-2}{k-1}^{-1}\right\}_{n=1}^{\infty}$. Then if $(\sum^{k}A)_c=B=(b_{pq})$, we note from (1) and (3) of Lemma 1 that B has convergent columns and there exists L such that $|b_{pq}|<L$, $p, q=1, 2, 3, \ldots$. Hence if $\sum^{-1}B=D=(d_{pq})$, then D has convergent columns and $\left|\sum_{q=1}^{n} d_{pq}\right|<L$, $n, p=1, 2, 3, \ldots$. By Theorem 1 of [3] there exists an increasing sequence α of positive integers such that if y is a null sequence and $F(y,\alpha)=w$, then Dw converges. Let y be an alternating null sequence such that $\{|y_p|\}$ is decreasing and $y\notin BV_1^*$. Let $F(y,\alpha)=w$. We note that $(\sum^{k}A)(c\Delta w)$ converges, since by 1) of Lemma 2, we have

$$Dw=(\textstyle\sum^{1}D)(\Delta w)=B(\Delta w)=(\sum^{k}A)_c(\Delta w)=(\sum^{k}A)(c\Delta w).$$

Let $s_p^{(n)}=\binom{p+k-n-1}{k-1}c_p$, $p=n, n+1, n+2, \ldots$. Then $\{s_p^{(n)}\}_{p=1}^{\infty}$ is a non-decreasing sequence convergent to 1, since

$$s_p^{(n)}=\prod_{q=2}^{k}\left(1-\frac{n-1}{p+k-q}\right).$$

Thus the series $\sum_{p=n}^{\infty} s_p^{(n)}\Delta w_p$ converges since $\sum\Delta w_p$ converges. Let

$$x_n=\sum_{p=n}^{\infty} s_p^{(n)}\Delta w_p,\quad n=1, 2, 3, \ldots.$$

We wish to show that $\{x_p\}$ is a null sequence. Let $d_p^{(n)}=1-s_p^{(n)}$, $p=n, n+1, n+2, \ldots$. Then $\{d_p^{(n)}\}_{p=n}^{\infty}$ is a nonincreasing null sequence. Let

$$w_n'=\textstyle\sum_{p=n}^{\infty} d_p^{(n)}\Delta w_p,\quad n=1, 2, 3, \ldots.$$

We note that w_n' can be written as an alternating series which satisfies the hypothesis of the alternating series test. Hence if Δw_m is the first nonzero Δw_p, $p\geq n$, then $|w_n'|<|d_m^{(n)}\Delta w_m|$. Thus $w_n'\to 0$ as $n\to\infty$. Clearly

$$w_n=\textstyle\sum_{p=n}^{\infty}\Delta w_p\to 0 \text{ as } n\to\infty.$$

Thus $x_n\to 0$ as $n\to\infty$, since $x_q=w_q-w_q'$, $q=1, 2, 3, \ldots$.

If n is a positive integer, then

$$\Delta x_n = \sum_{p=n}^{\infty} \binom{p+k-n-1}{k-1} c_p \Delta w_p - \sum_{p=n+1}^{\infty} \binom{p+k-n-2}{k-1} c_p \Delta w_p$$

$$= \binom{k-1}{k-1} c_n \Delta w_n + \sum_{p=n+1}^{\infty} \left[\binom{p+k-n-1}{k-1} - \binom{p+k-n-2}{k-1} \right] c_p \Delta w_p$$

$$= \sum_{p=n}^{\infty} \binom{p+k-n-2}{k-2} c_p \Delta w_p .$$

Similarly it can be shown that

$$\Delta^r x_n = \sum_{p=n}^{\infty} \binom{p+k-n-r-1}{k-r-1} c_p \Delta w_p, \quad r = 1, 2, \dots, k-1.$$

Thus $\Delta^k x_n = c_n \Delta w_n$.

If $1 \leqq r \leqq k-2$, let

$$u_n^{(r)} = \sum_{p=n}^{\infty} \frac{n^r}{(p+r-1)(p+r-2)\cdots p} \Delta w_p,$$

$$v_p^{(r,n)} = 1 - \prod_{q=2}^{k-r} \left(1 - \frac{n+r-1}{p+k-q}\right), \quad p \geqq n,$$

and

$$t_n^{(r)} = \sum_{p=n}^{\infty} \frac{v_p^{(r,n)} n^r}{(p+r-1)(p+r-2)\cdots p} \Delta w_p .$$

We note that $u_n^{(r)}$ and $t_n^{(r)}$ can be written as alternating series which satisfy the hypothesis of the alternating series test. Hence if Δw_m is the first nonzero Δw_p, $p \geqq n$, then

$$| u_n^{(r)} | < \frac{n^r}{(m+r-1)(m+r-2)\cdots m} | \Delta w_m |,$$

and

$$| t_n^{(r)} | < \frac{v_m^{(r,n)} n^r}{(m+r-1)(m+r-2)\cdots m} | \Delta w_m | .$$

Since

$$n^r \Delta^r x_n = \frac{(k-1)!}{(k-r-1)!} \sum_{p=n}^{\infty} \frac{(1 - v_p^{(r,n)}) n^r}{(p+r-1)(p+r-2)\cdots p} \Delta w_p, \quad r = 1, 2, \dots, k-2,$$

we have

$$| n^r \Delta^r x_n | = | u_n^{(r)} - t_n^{(r)} | \frac{(k-1)!}{(k-r-1)!}$$

$$< \frac{n^r (1 + v_m^{(r,n)})}{(m+r-1)(m+r-2)\cdots m} | \Delta w_m | \frac{(k-1)!}{(k-r-1)!}$$

$$< 2 | \Delta w_m | \frac{(k-1)!}{(k-r-1)!} .$$

Hence $n^r \Delta^r x_n \to 0$ as $n \to \infty$, $r = 1, 2, \dots, k-2$. Also, it is easy to show that $n^{k-1} \Delta^{k-1} x_n \to 0$ as $n \to \infty$.

Recalling that $(\Sigma^k A)(c\Delta w)$ converges and that $(\Sigma^k A)(c\Delta w) = (\Sigma^k A)(\Delta^k x)$, we see that Ax converges, since by Lemma 2,

$$Ax = \left(\sum^r A\right)(\Delta^r x), \quad r = 1, 2, \ldots, k.$$

We have

$$\Delta^k x_n = c_n \Delta w_n = \binom{n+k-2}{k-1}^{-1} \Delta w_n,$$

and so

$$\sum_{n=1}^{\infty} \binom{n+k-2}{k-1} |\Delta^k x_n| = \sum_{n=1}^{\infty} |\Delta w_n| = \sum_{p=1}^{\infty} |\Delta y_p| = \infty,$$

since $y \notin BV_1^*$. Thus $x \notin BV_k^*$. This completes the proof of the theorem.

Remark. If α is an increasing sequence of positive integers and y is an alternating null sequence such that $\{|y_p|\}$ is decreasing, let $G(y, \alpha)$ denote the sequence x (if it exists) determined in the proof of Theorem 1. The proof of Theorem 1 can be modified slightly to prove the following statement:

If A sums every null sequence in BV_k^* and α is an increasing sequence of positive integers, then there exists a subsequence β of α such that if γ is an increasing sequence of positive integers such that for some m, $\{\gamma_p\}_{p=m}^{\infty}$ is a subsequence of β, y is an alternating null sequence such that $\{|y_p|\}$ is decreasing, and $G(y, \gamma) = x$, then Ax converges.

Analogous to Theorem 2 of [2], we have the following result which we state without proof.

Theorem 2. If M is a countable set of matrices, each of which sums every null sequence in BV_k^*, then there exists a null sequence not in BV_k^* which every matrix in M sums.

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