PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 12 (26), 1971. pp. 31—35

ON MULTIHOMOMORPHISMS

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(Communicated Oktober 27, 1971)

1. Let $(X, \cdot)$ and $(Y, \circ)$ be two arbitrary groups. Let us consider the mult:-valued mapping of X into Y, i.e. the correspondence between elements of X and non-void subsets of Y. In this paper we study such multi-valued mapping f of the group X into the group Y which satisfy the relation:

$$f(x \cdot y) = f(x) \circ f(y) \tag{1}$$

where $f(x) \circ f(y) = \{a \circ b : a \in f(x),\ b \in f(y)\}$.

Notice that this multiplication of subsets of a group is associative, that is $(A \circ B) \circ C = A \circ (B \circ C)$ for any non-void $A, B, C \subset Y$.

A multi-valued mapping f satisfying the relation (1) will be called a *multihomomorphism.*

A reason for studying mappings of the form (1) is a problem raised by J. Aczél (see [1], p. 380).

2. Suppose that f be a multihomomorphism of the group X into the group Y. Then from (1) we obtain the following useful consequences:

$$f(x) = f(x) \circ f(e) = f(e) \circ f(x) \tag{2.1}$$

$$f(e) = f(x) \circ f(x^{-1}) = f(x^{-1}) \circ f(x) \tag{2.2}$$

$$f(e) = f(e) \circ f(e) \qquad (e = \text{identity element of } X). \tag{2.3}$$

The relation (2.3) means that the image of the identity element of the group X is a subset of Y closed under the operation „$\circ$", and consequently is a semi-group.

If $\bar{f}$ is a homomorphism of a group X into a quotient group Y/A of the group Y (where A is a normal subgroup of Y), then the multi-valued mapping f of X into Y defined by $f(x) = \bar{f}(x) \subset Y$ is a multihomomorphism.

At this point we shall prove that under some conditions the converse is also true, i.e. that to every multihomomorphism f of a group X into a group Y we can correspond a single-valued homomorphism $\bar{f}$ of X into $f(X)/A$ (A is a normal subgroup of $f(X)$) such that $\bar{f}(x) = f(x)$.

First of all we shall make the following

S u p p o s i t i o n 1. $f(e)$ is a subgroup of Y.

Under the supposition 1. we have the following lemmas.

Lemma 1. *Let f be a multihomomorphism of the group X into the group Y. If for some $x \in X$, $f(x) \cap f(e) \neq \emptyset$, then $f(x) = f(e) = f(x^{-1})$.*

Proof. Let $f(x) \cap f(e) \neq \emptyset$. Then there exists $a \in f(x) \cap f(e)$. From (2.1) we have

$$f(x) = a \circ f(e) \cup (f(x) \setminus \{a\}) \circ f(e). \tag{2.4}$$

Since $f(e)$ is a subgroup of Y and $a \in f(e)$ so that $a \circ f(e) = f(e)$, (2.4) implies

$$f(x) \supset f(e). \tag{2.5}$$

On the other hand, using (2.2), (2.5) and (2.1), we obtain

$$f(e) = f(x) \circ f(x^{-1}) \supset f(e) \circ f(x^{-1}) = f(x^{-1}),$$

which together with (2.5) gives

$$f(x^{-1}) \subset f(e) \subset f(x). \tag{2.6}$$

From (2.6) it follows that $f(x^{-1}) \cap f(e) \neq \emptyset$ and exchanging the rolls of x and x^{-1} we have the converse inclusion

$$f(x) \subset f(e) \subset f(x^{-1}). \tag{2.7}$$

From (2.6) and (2.7) it follows the above assertion.

Lemma 2. *Let f be a multihomomorphism of a group X into a group Y. If $a \in f(x)$, then $a^{-1} \in f(x^{-1})$.*

Proof. Let a be an arbitrary element of $f(x)$, and b some fixed element of $f(x^{-1})$. Then

$$a \circ b = c \in f(x) \circ f(x^{-1}) = f(e).$$

Since $f(e)$ is a group $c^{-1} \in f(e)$. But

$$a^{-1} = b \circ c^{-1} \in f(x^{-1}) \circ f(e) = f(x^{-1})$$

and lemma is proved.

Lemma 3. *Let f be a multihomorphism of a group X into a group Y. If $f(x) \cap f(y) \neq \emptyset$, then $f(x) = f(y)$.*

Proof. If $f(x) \cap f(y) \neq \emptyset$, then there exist some $a \in f(x) \cap f(y)$. Since $a \in f(x)$ and $a \in f(y)$, according to lemma 2. it follows that $a^{-1} \in f(y^{-1})$. Then (denoting by e_1 the identity element of Y) we have

$$a \circ a^{-1} = e_1 \in f(x) \circ f(y^{-1}) = f(xy^{-1}).$$

Since $e_1 \in f(e)$ it follows that $f(xy^{-1}) \cap f(e) \neq \emptyset$, and according to lemma 1.

$$f(xy^{-1}) = f(e),$$

i.e.

$$f(x) \circ f(y^{-1}) = f(e). \tag{2.8}$$

From (2.1) and (2.8) it follows

$$f(y) = f(e) \circ f(y) = f(x) \circ f(y^{-1}) \circ f(y) = f(x) \circ f(e) = f(x).$$

Lemma is proved.

L e m m a 4. (*on representation*). *If $f(e)$ is a subgroup of Y, then for every $x \in X$, the subset $f(x)$ of Y is of the form*

$$f(x) = a \circ f(e) = f(e) \circ a \tag{2.9}$$

where a is an arbitrary element of $f(x)$.

P r o o f. From $a \in f(x)$, it follows $\{a\} \subset f(x)$. Then, using (2.1) we obtain

$$f(x) = f(x) \circ f(e) \supset a \circ f(e) \tag{2.10}$$

(instead of $\{a\} \circ f(e)$ we write $a \circ f(e)$).

To prove the converse inclusion, let us take an arbitrary element b of $f(x)$. Since Y is a group, there exist some element t of Y such that

$$b = a \circ t. \tag{2.11}$$

We aim to prove that $t \in f(e)$.

Since $a \in f(x)$, according to lemma 2. $a^{-1} \in f(x^{-1})$, and hence from (2.11) it follows

$$t = a^{-1} \circ b \in f(x^{-1}) \circ f(x) = f(e).$$

So, every element b of $f(x)$ is of the form (2.11) and we have

$$f(x) \subset a \circ f(e). \tag{2.12}$$

From (2.10) and (2.12) we obtain $f(x) = a \circ f(e)$.

The relation $f(x) = f(e) \circ a$ is provable in the same way.

P r o p o s i t i o n 2.1. *The set $f(X) = \cup \{f(x) : x \in X\}$ is a subgroup of Y, provided that $f(e)$ is a subgroup of Y.*

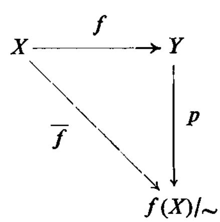
P r o o f. Let $a, b \in f(X)$. Then there exist $x, y \in X$ such that $a \in f(x)$, $b \in f(y)$, so that $a \circ b \in f(x) \circ f(y) = f(xy) \subset f(X)$. If $a \in f(x)$, from lemma 2. we conclude that $a^{-1} \in f(x^{-1})$, hence $a^{-1} \in f(X)$.

P r o p o s i t i o n 2.2. *If $f(e)$ is a subgroup of Y, that $f(e)$ is a normal subgroup of $f(X) \subset Y$.*

P r o o f. It follows from lemma 4.

T h e o r e m 1. *Let $f: X \to Y$ be a multihomomorphism of a group X into group Y and $f(e)$ a subgroup of Y. Then $f(e)$ is a normal subgroup of $f(X)$ and there exists a single-valued homomorphism $\bar{f}$ of X into the quotient group $f(X)/f(e)$ such that $f(x) = \bar{f}(x)$ for every $x \in X$.*

In other words the following diagram commutes:



$\bar{f} = pf$ $(a \sim b \Leftrightarrow a \circ b^{-1} \in f(e))$.

Proof follows from lemma 3. and proposition 2.1. and 2.2.

The following example shows that the supposition 1. is essential for the conclusion of Theorem 1.

Example 2.1. Let $X=Y=R$ the additive group of reals and $f: R\rightarrow R$ is defined by:

$$f(0)=R^{+}\cup\{0\} \quad \text{(the set of reals } x\geqslant 0)$$

and

$$f(x)=ax+f(0),$$

where a is some real number. The multi-valued mapping f so defined is a multihomomorphism of R into R, but the conclusion of Theorem 1. is not true. Notice that $f(0)$ is not a subgroup of R.

Theorem 2. *Let f be a multihomomorphism of a group X into a group Y and let $f(X)$ be a periodic subgroup of Y. Then the conclusion of Theorem 1. is valid.*

Proof. Only to be proved is that under above hypothesis $f(e)$ is a subgroup of Y. But, since, $f(e)$ a subsemi-group of $f(X)$, this fact is directly seen.

Corollary: To every multihomomorphism f of a group X into a finite group Y, one can correspond a single-valued homomorphism $\bar{f}$ of X into some quotient group of $f(X)$ (more precisely to $f(X)/f(e)$) such that for every $x\in X$ $\bar{f}(x)=f(x)$.

So multihomomorphism of a group X into some finite group Y are always reduced (in some sense) to single-valued homomorphisms.

3. The representation lemma 4. suggests the following.

Proposition 3.1. *Let $h: X\rightarrow Y$ be a homomorphism of a group X into a group Y and A a normal subgroup of $h(X)$. Then the multi-valued mapping f of X into Y defined for every $x\in X$ by*

$$f(x)=h(x)\circ A \tag{3.1}$$

is a multihomomorphism.

Proof. We have, since A is a normal subgroup of $h(X)$ and the operation „∘" is associative,

$$\begin{aligned} f(x)\circ f(y) &= (h(x)\circ A)\circ(h(y)\circ A) \\ &= h(x)\circ h(y)\circ A = h(xy)\circ A = f(xy). \end{aligned}$$

The converse of the proposition 3.1. is not true in general, as the following example shows.

Example 3.1. Let $X=\{e, a\}$ be the cyclic group of order two and Y the additive group of integers. Then multi-valued mapping of X into Y defined by

$f(e)=2E$ (= the set of even integers and 0) and

$f(a)=2E+1$ (= the set of all odd integers).

Then f is a multihomomorphism, but there is not homomorphism h of X into Y such that f be of the form (3.1).

This means that the fact that $f(e)$ is a subgroup of X does not guarantee that every multihomomorphism of X into Y is of the form (3.1).

Problem. Under which conditions imposed on the groups $X, Y, f(e)$ the converse of the proposition 3.1. is valid?

Let f be a multihomomorphism of a group X into a group Y. If there exists a single-valued homomorphism h of X into Y satisfying the relation (3.1) we shall call it a selective homomorphism of the multihomomorphism f.

In the following we suppose that for cosidered multihomomorphism f there exists a selective homomorphism h. In a special case the converse of proposition 3.1. is also true.

Theorem 3. *Let the group Y be the direct product of two subgroups S and T, then every multihomomorphism of X into Y satisfying the Supposition 1. that is $f(e)=T$ is of the form*

$$f(x)=h(x)\circ f(e)$$

where $h: X\to Y$ is a single-valued homomorphism. In other words, under the above conditions, every multihomomorphism admits a selective homomorphism.

Proof. Since $f(e)$ is a subgroup of Y, according to Lemma 4. we have the following representation $f(x)=a_x\circ f(e)$. Y being the direct product of S and $T=f(e)$ $(S\cap T=\{e_1\})$, we conclude that $a_x\in S$ or $f(x)=f(e)$. If $f(x)=f(e)$ we put $a_x=e_1$.

Since f is a multihomomorphism from (1) we have

$$a_{xy}\circ f(e)=a_x\circ f(e)\circ a_y\circ f(e). \tag{3.2}$$

$f(e)$ being a normal subgroup of Y, we obtain

$$a_{xy}\circ f(e)=a_x\circ a_y\circ f(e). \tag{3.3}$$

Let us prove that (3.3) under above conditions implies

$$a_{xy}=a_x\circ a_y. \tag{3.4}$$

Suppose that $a\circ f(e)=b\circ f(e)$, then there exist $m, n\in f(e)$ such that

$$a\circ m=b\circ n. \tag{3.5}$$

From (3.5) follows

$$b^{-1}\circ a\in S\cap T. \tag{3.6}$$

Since $S\cap T=\{e_1\}$, we obtain $b^{-1}\circ a=e_1$ or $a=b$, which proves the assertion, and (3.4) is also proved.

So mapping $h: X\to Y$ defined by $h(x)=a_x$ is a single-valued homomorphism of X into Y more precisely of X into Y.

Theorem is proved.

Remarque. Conditions of Theorem 3. guarantee the existence of a selective homomorphism for every multihomomorphism of X into Y, but those conditions are not only ones, under which a selective homomorphism exists, as it is easily seen from examples. Thus the above problem is not answered completely.

REFERENCE

[1] J. Aczél, *Lectures on Functional Equations and Their Applications*, New York and London, 1966.