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A CERTAIN CLASS OF MAPPINGS IN TOPOLOGICAL SPACES

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Introduction. Let (M, d) be a metric space and let T be a mapping of M into itself. The mapping T is noncontractive if $d(Tx, Ty) \geq d(x, y)$, nonexpansive if $d(Tx, Ty) \leq d(x, y)$ and isometric if $d(Tx, Ty) = d(x, y)$ holds for every $x, y \in M$. If $\varepsilon > 0$ and $d(x, Tx) \leq \varepsilon$ for all $x \in S \subseteq M$, then T is an ε-mapping on S.

In [2] noncontractive, nonexpansive and isometric mappings of totally bounded metric spaces into itself are investigated. It is the purpose of this paper to point out that some results from [2] can be extended to the class of topologically totally bounded spaces.

1. Definitions. Let (E, τ) be a uniformizable topological space and let $\mathcal{R}(E)$ be the family of all uniformities $\mathcal{U}$ on E such that the uniform topology of $(E, \mathcal{U})$ is equivalent to the original topology.

1.1. A topological space E is *topologically totally bounded* if there exists $\mathcal{U} \in \mathcal{R}(E)$ such that the uniform space $(E, \mathcal{U})$ is totally bounded ($(E, \mathcal{U})$ is totally bounded if for every $V \in \mathcal{U}$ there exists a finite subset A of E such that $V[A] = E$).

The set of all uniformities $\mathcal{U}$ on E for which $(E, \mathcal{U})$ is totally bounded we shall denote by $\mathcal{TB}(E)$.

1.2. Let $(E, \mathcal{U})$ be a uniform space and let $\mathcal{B}$ be a basis for the uniformity $\mathcal{U}$. The basis $\mathcal{B}$ is said to be *ample* if whenever $(x, y) \in U \in \mathcal{B}$, there exists a $V \in \mathcal{B}$ for which $(x, y) \in V \subset \bar{V} \subset U$.

In addition we introduce the following definitions:

1.3. A mapping $T: E \to E$ is *topologically noncontractive* if there exists a basis $\mathcal{B}$ for a uniformity $\mathcal{U} \in \mathcal{R}(E)$ such that

(1) $$(Tx, Ty) \in V \Rightarrow (x, y) \in V$$

for every $V \in \mathcal{B}$.

1.4. A mapping $T: E \to E$ is *topologically nonexpansive* if there exists a basis $\mathcal{B}$ for a uniformity $\mathcal{U} \in \mathcal{R}(E)$ such that

(2) $$(x, y) \in V \Rightarrow (Tx, Ty) \in V$$

for every $V \in \mathcal{B}$.

1.5. A mapping $T: E \to E$ is a *topological isobasism* if there exists a basis $\mathcal{B}$ for one $\mathcal{U} \in \mathcal{R}(E)$ such that

(3) $$(x, y) \in V \Leftrightarrow (Tx, Ty) \in V$$

for every $V \in \mathcal{B}$.

1.6. Let $\mathcal{U} \in \mathcal{R}(E)$ and let $U \in \mathcal{U}$ be arbitrary. A mapping $T: E \to E$ is a *U-mapping on a subset* $S \subseteq E$ if

$$Tx \in U[x] \tag{4}$$

holds for every $x \in S$.

2. Theorems. In [2] the following result is obtained:

2.1. Theorem. *Let T be a noncontractive mapping of a totally bounded metric space M into itself. Then for any $\varepsilon > 0$ and every finite subset F of M there exists a positive integer $m = m(\varepsilon, F)$, such that the mapping T^m of M into M is an ε-mapping on F.*

The following theorem is a generalization of theorem 2.1.

2.2. Theorem. *Let E be a topologically totally bounded Hausdorff space and let T be a mapping of E into itself.*

If $\mathcal{U} \in \mathcal{TB}(E)$ is any uniformity for E, $\mathcal{B}$ is a basis for $\mathcal{U}$ and T satisfies (1) *relative to $\mathcal{B}$, then for an arbitrary $U \in \mathcal{U}$ and every finite subset F of E there exists a positive integer $m = m(U, F)$ such that the mapping T^m is a U-mapping on F.*

Proof. Let $U \in \mathcal{U}$ be arbitrary and let $V, W, Z \in \mathcal{B}$ be such that $W \subset U$, $Z \circ Z \subset W$ and $V \subset Z \cap Z^{-1}$. Let $F = \{x_1, x_2, \ldots, x_p\}$ be an arbitrary finite subset of E.

Consider the sequence

$$Tx_1, T^2x_1, \ldots, T^nx_1, \ldots .$$

By assumption the space $(E, \mathcal{U})$ is totally bounded, thus $E = V[A]$ for some finite subset A of E. Then there must exist an element in A, denote it a_1, such that the set $V[a_1]$ contains infinite number of members of the sequence $\{T^nx_1 : n \in N\}$. Thus

$$T^nx_1 \in V[a_1] \qquad \text{for all } n \in N_1,$$

where N_1 is an infinite subset of the natural number N. So we have

$$(a_1, T^nx_1) \in V \qquad \text{for all } n \in N_1 \subseteq N.$$

Consider now the sequence $\{T^nx_2 : n \in N_1\}$. There must exist an element in A, call it a_2, such that $V[a_2]$ contains infinite number of members of $\{T^nx_2 : n \in N_1\}$. Thus

$$(a_2, T^nx_2) \in V \qquad \text{for all } n \in N_2 \subseteq N_1 \subseteq N.$$

Continuing this process we obtain p sequences such that:

if $n \in N_1 \subseteq N$, then $(a_k, T^nx_k) \in V$ for $k = 1$,

if $n \in N_2 \subseteq N_1 \subseteq N$, then $(a_k, T^nx_k) \in V$ for $k = 1, 2$,

. . . .

if $n \in N_p \subseteq N_{p-1} \subseteq \cdots \subseteq N$, then $(a_k, T^nx_k) \in V$ for $k = 1, 2, \ldots, p$.

Let $i, j \in N_p$ $(i < j)$ and let $m = j - i$. Then

$$(T^ix_k, T^jx_k) = (T^ix_k, a_k) \circ (a_k, T^ja_k) \in V^{-1} \circ V \subset Z \circ Z \subset W$$

holds for every $x_k \in F$, so we have

(5) $$(T^i x_k, T^i T^m x_k) = (T^i x_k, T^j x_k) \in W$$

for every $x_k \in F$.

Since $W \in \mathcal{B}$ and T is a noncontractive, it follows

$$(T^i x_k, T^j x_k) = (TT^{i-1} x_k, TT^{j-1} x_k) \in W \Rightarrow (T^{i-1} x_k, T^{j-1} x_k) \in W$$
$$\Rightarrow \cdots \Rightarrow (x_k, T^m x_k) \in W.$$

Thus

$$(x_k, T^m x_k) \in W \subset U,$$

namely

$$T^m x_k \in U[x_k]$$

for every $x_k \in F$, which was what we wanted to show.

Observe that with the same hypothesis as in this theorem, for any finite subset $F = \{x_1, x_2, \dots, x_p\}$ of E and every collection $\mathcal{O}_F = \{O_1, O_2, \dots, O_p\}$ of neighborhoods of points $x_1, x_2, \dots, x_p \in F$ there is a positive integer $m = m(F, \mathcal{O}_F)$ such that $T^m x_k \in O_k$ $(k = 1, 2, \dots, p)$. Indeed, if $U_1, U_2, \dots, U_p \in \mathcal{U}$ are such that $U[x_k] \subset O_k$ $(k = 1, 2, \dots, p)$ and $U = \cap\{U_k : k = 1, 2, \dots, p\}$, then choosing $V, W, Z \in \mathcal{B}$ as above and applying the same method of operation, we obtain $m(F, \mathcal{O}_F) \leqslant m(F, W)$.

If T is a topologically isobasic mapping then it may be proved more than in the above theorems. In [2] we proved the following:

2.3. Theorem. *Let T be an isometry on a totally bounded metric space M and let S be any subset of M. Then for any $\varepsilon > 0$ there is a positive integer $m = m(\varepsilon, S)$, such that the mapping T^m is an ε-mapping on S.*

The following theorem is a generalization of the above theorem.

2.4. Theorem. *Let E be a topologically totally bounded Hausdorff space and let T be a mapping of E into itself. If $\mathcal{U} \in \mathcal{TB}(E)$ is any uniformity for E, $\mathcal{B}$ is a basis for $\mathcal{U}$ and T satisfies (3) relative to $\mathcal{B}$, then for an arbitrary $U \in \mathcal{U}$ and any subset $S \subseteq E$ there exists a positive integer $m = m(U, S)$ such that the mapping T^m is a U-mapping on S.*

Proof. Let $V, W \in \mathcal{B}$ be such that $W \circ W \circ W \subset U \in \mathcal{U}$ and $V \subset W \cap W^{-1}$. Since S is a subset of the totally bounded space $(E, \mathcal{U})$, it follows that $S \subseteq V[B]$ for some finite subset B of E. By definition of B, for every $x \in S$ there exists an element of B, denote it $b(x)$, such that $x \in V[b(x)]$.

According to Theorem 2.2., there is a positive integer $m = m(V, B)$ such that T^m is a V-mapping on B, i.e, such that

(6) $$(b, T^m b) \in V$$

for every $b \in B$.

Now we shall show that T^m is a U-mapping on S. Let $x \in S$ be arbitrary and let $b(x)$ be an element of B such that

(7) $$(b(x), x) \in V.$$

Since T is an isobasic mapping relative to $\mathcal{B}$, T^m is also an isobasic mapping relative to the same basis. Thus from (7) it follows

(8) $$(T^m b(x), T^m x) \in V.$$

Since

$$(x, T^m x) = (x, b(x)) \circ (b(x), T^m b(x)) \circ (T^m b(x), T^m x),$$

from (6), (7) and (8) it follows

$$(x, T^m x) \in V^{-1} \circ V \circ V \subset W \circ W \circ W \subset U,$$

i.e.

$$T^m x \in U[x].$$

Since $x \in S$ was arbitrary, the theorem is proved.

2.5. Corollary. *Let E be a topologically totally bounded Hausdorff space and let T be a mapping of E onto itself. If $\mathcal{U} \in \mathcal{TB}(E)$ is any uniformity for E, $\mathcal{B}$ is an open, ample basis for $\mathcal{U}$ and T satisfies* (2) *relative to $\mathcal{B}$, then for arbitrary $U \in \mathcal{U}$ and any subset $S \subseteq E$ there exists a positive integer $m = m(U, S)$ such that the mapping T^m is a U-mapping on S.*

Proof. By result of T.Brown and W. Comfort [1, Theorem 2.1], it follows that T satisfies (3) relative to $\mathcal{B}$.

2.6. Corollary. *Let E be a compact Hausdorff space and let T be a mapping of E into itself. If $\mathcal{U} \in \mathcal{R}(E)$ is any uniformity for E, $\mathcal{B}$ is an open, ample basis for $\mathcal{U}$ and T satisfies* (1) *relative to $\mathcal{B}$, then for arbitary $U \in \mathcal{U}$ and any subset $S \subseteq E$ there is a positive integer $m = m(U, S)$ such that the mapping T^m is a U-mapping on S.*

Proof. From the cited paper of T. Brown and W. Comfort, (Corollary 3.7), it follows that T satisfies (3) relative to $\mathcal{B}$.

REFERENCES

[1] Brown T., Comfort W., *New methods for expansion and contraction maps in uniform spaces*, Proc. Amer. Math. Soc. 11 (1960), 483—486.

[2] Ćirić Lj., *O jednoj klasi preslikavanja u metričkim prostorijama*, Mat. vesnik 7 (22), 1970, 159—164.