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GENERALIZED CONTRACTIONS AND FIXED-POINT THEOREMS

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1. Introduction. Let (M, d) be a metric space and let T be a mapping of M into itself. A mapping T is a contraction if there exists a number q, $0 \leq q < 1$, such that the condition

$$d(Tx, Ty) \leq q \cdot d(x, y) \tag{A}$$

holds for every $x, y \in M$.

The well-known Banach contraction principle is the following:

If $T: M \to M$ is a contraction mapping of a complete metric space M into itself, then

1° *There is in M a unique fixed-point u under T,*

2° *$T^n x \to u$ for every $x \in M$ and*

3° $d(T^n x, u) \leq \dfrac{q^n}{1-q} \cdot d(x, Tx)$.

The theorem of Banach and its extensions usualy are proved by the fact that the geometrical series $\sum_{n=0}^{\infty} q^n$ is convergent. Some different proof of the Banach theorem is given by R. Kannan [3, theorem 3], where he investigated properties of subsets of M, defined as $S_a = \{x \in M : d(x, Tx) \leq a\}$, $0 < a < +\infty$. Further, in [4, p. 406] he showed the following:

If M is a complete metric space and mapping $T: M \to M$ is such that the condition

$$d(Tx, Ty) \leq a(d(x, Tx) + d(y, Ty)); \qquad 0 < a < \frac{1}{2} \tag{B}$$

holds for every $x, y \in M$, then T leaves exactly one point of M fixed.

The conditions (A) and (B) are independent, as it was shown by two examples in [4].

The purpose of this note is to define and to investigate a class of generalized contractions, which includes the Banach's contractions and the mappings which satisfy (B). Every mapping of this class is such that we can obtain all three good properties of the Banach's contraction principle.

2. Let (M, d) be a metric space and let T be a mapping of M into itself.

2.1. Definition. *A mapping* $T: M \to M$ *is said to be a* λ*-generalized contraction iff for every* $x, y \in M$ *there exist non-negative numbers* $q(x, y)$, $r(x, y)$, $s(x, y)$ *and* $t(x, y)$ *such that*

$$\sup_{x, y \in M} \{q(x, y) + r(x, y) + s(x, y) + 2t(x, y)\} = \lambda < 1$$

and

$$\text{(C)} \qquad d(Tx, Ty) \leqslant q(x, y)\, d(x, y) + r(x, y)\, d(x, Tx) + s(x, y)\, d(y, Ty) + \\ + t(x, y)\,(d(x, Ty) + d(y, Tx))$$

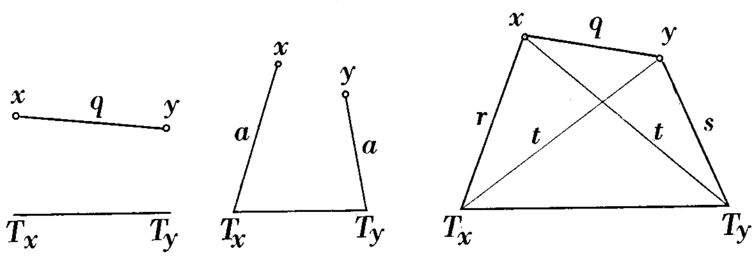
holds for every $x, y \in M$.

It is easy to see that:

1) If T satisfies the condition (A), then T also satisfies the condition (C) for every $r(x, y)$, $s(x, y)$ and $t(x, y)$ such that $0 \leqslant r(x, y) + s(x, y) + 2t(x, y) < 1 - q - c$ $(0 < c \leq 1 - q)$, and

2) If T satisfies (B), then T also satisfies (C) for every $q(x, y)$ such that $0 \leq q(x, y) < 1 - 2a - c$ $(0 < c \leq 1 - 2a)$.

Geometrical:



a) b) c)

Now we shall give two examples which show that there exist such mappings which do not satisfy (A) or (B), but satisfy the condition (C).

Example 1. Let $M = [0, 2] \subset R^1$ and let $Tx = \frac{x}{9}$ for $0 \leq x \leq 1$; $Tx = \frac{x}{10}$ for $1 < x \leq 2$. The mapping T does not satisfy (A) on whole M. For example, if $x = \frac{999}{1000}$ and $y = \frac{1001}{1000}$, then

$$d(Tx, Ty) = \frac{981}{90000} > 5 \cdot \frac{180}{90000} = 5d(x, y).$$

But T satisfies (C) with $q(x, y) \equiv \frac{1}{10}$, $r(x, y) = s(x, y) \equiv \frac{1}{4}$ and $t(x, y) \equiv \frac{1}{6}$ for all $x, y \in M$.

Example 2. Let $M = [0, 10] \subset R^1$ and let $Tx = \frac{3}{4}x$ for all $x \in M$. If, for example, $x = 0$ and $y = 8$, then T does not satisfy (B) for $a < 3$. But T satisfies (C) on whole M with $q(x, y) \equiv \frac{3}{4}$, $r(x, y) = s(x, y) = t(x, y) \equiv \frac{1}{20}$.

The following definitions in [2] are introduced:

2.2. Definition. *Let T maps M into M; a space M is said to be T-orbitally complete if every sequence $\{T^{n_i}x : i \in N\}$, $x \in M$, which is a Cauchy sequence, has a limit point in M.*

It is obvious that if M is a complete space, then M is T-orbitally complete for any mapping $T : M \to M$. In [2, example 3] it was shown that noncomplete metric space may be orbitally complete relative to one mapping, but not to another.

2.3. Definition. *A mapping T of a space M into itself is said to be orbitally continuous if $u \in M$ and such that $u = \lim_{i\to\infty} T^{n_i}x$ for some $x \in M$, then $Tu = \lim_{i\to\infty} TT^{n_i}x$.*

2.4. Definition. *A mapping $T : (M, d) \to (M, d)$ is said to be a contraction type mapping if for every $x, y \in M$ there are numbers $q(x, y)$, $0 \leq q(x, y) < 1$ and $\delta(x, y) > 0$ such that*

$$d(T^n x, T^n y) \leq (q(x, y))^n \cdot \delta(x, y)$$

holds for every $n = 1, 2, \ldots$

2.5. Theorem. *Let T be a λ-generalized contraction of T-orbitally complete metric space M into itself. Then*

(i) *There is in M a unique fixed-point u under T,*

(ii) *$T^n x \to u$ for every $x \in M$ and*

(iii) $d(T^n x, u) \leq \dfrac{\lambda^n}{1-\lambda} \cdot d(x, Tx)$,

Proof. Let $x \in M$ be arbitrary and define a sequence.

$$x_0 = x,\ x_1 = Tx_0, \ldots, x_n = Tx_{n-1} = T^n x_0, \ldots \tag{1}$$

Since T is a λ-generalized contraction, by (C) and definition of the sequence (1) it follows

$$\begin{aligned} d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) &\leq q(x_{n-1}, x_n)\, d(x_{n-1}, x_n) + \\ &+ r(x_{n-1}, x_n)\, d(x_{n-1}, Tx_{n-1}) + s(x_{n-1}, x_n)\, d(x_n, Tx_n) + \\ &+ t(x_{n-1}, x_n)\,(d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})) = \\ &= q(x_{n-1}, x_n)\, d(x_{n-1}, x_n) + r(x_{n-1}, x_n)\, d(x_{n-1}, x_n) + \\ &s(x_{n-1}, x_n)\, d(x_n, x_{n+1}) + t(x_{n-1}, x_n)\, d(x_{n-1}, x_{n+1}). \end{aligned}$$

Applying the triangle inequality we get

$$\begin{aligned} d(x_n, x_{n+1}) \leq (q(x_{n-1}, x_1) &+ r(x_{n-1}, x_n))\, d(x_{n-1}, x_n) + \\ &+ s(x_{n-1}, x_n)\, d(x_n, x_{n+1}) + t(x_{n-1}, x_1)\,(d(x_{n-1}, x_n) + \\ &+ d(x_n, x_{n+1})). \end{aligned}$$

Consequently

$$d(x_n, x_{n+1}) \leq \frac{q(x_{n-1}, x_n) + r(x_{n-1}, x_n) + t(x_{n-1}, x_n)}{1 - s(x_{n-1}, x_n) - t(x_{n-1}, x_n)}\, d(x_{n-1}, x_n) \tag{2}$$

Since $\lambda < 1$, from

$$q(x, y) + r(x, y) + t(x, y) + \lambda s(x, y) + \lambda t(x, y) \leq \lambda$$

we have that

$$\frac{q(x, y) + r(x, y) + t(x, y)}{1 - s(x, y) - t(x, y)} \leqslant \lambda$$

holds for all $x, y \in M$. Thus from (2) it follows

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n), \tag{3}$$

which shows that a generalized contraction is a contraction for certain pair of points.

Repeating this argument n-times we obtain

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n) \leq \cdots \leq \lambda^n d(x, Tx).$$

Then for any positive integer p we have

$$d(x_n, x_{n+p}) \leq \sum_{i=1}^{p} d(x_{n+i-1}, x_{n+i}) \leq \sum_{i=1}^{p} \lambda^{n+i-1} d(x, Tx).$$

Therefore

$$d(x_n, x_{n+p}) \leq \frac{\lambda^n}{1-\lambda} d(x, Tx). \tag{4}$$

Since $\lambda < 1$, $\lambda^n \to 0$ when $n \to \infty$. So from (4) it follows that the sequence (1) is Cauchy. Since M is T-orbitally complete, there is a point $u \in M$ such that

$$u = \lim_{n \to \infty} T^n x = \lim_{n \to \infty} x_n. \tag{5}$$

Now we shall show that

$$Tu = \lim_{n \to \infty} Tx_n = \lim_{n \to \infty} x_{n+1} = u \tag{6}$$

i.e., that u is a fixed point under T.

Since T is the generalized contraction, by (C) and the triangle inequality it follows

$$\begin{aligned} d(Tu, Tx_n) &\leq q(u, x_n) d(u, x_n) + r(u, x_n)(d(u, x_{n+1}) + d(x_{n+1}, Tu)) + \\ &\quad + s(u, x_n) d(x_n, x_{n+1}) + t(u, x_n)(d(u, x_{n+1}) + d(Tu, x_n)) \leqslant \\ &\leqslant \lambda d(u, x_n) + (r(u, x_n) + t(u, x_n)) d(u, x_{n+1}) + \\ &\quad + r(u, x_n) d(Tx_n, Tu) + s(u, x_n) d(x_n, x_{n+1}) + \\ &\quad + t(u, x_n)(d(Tu, Tx_n) + d(Tx_n, x_n)) \leq \\ &\leq \lambda d(u, x_n) + \lambda d(u, x_{n+1}) + \\ &\quad + (r(u, x_n) + t(u, x_n)) d(Tu, Tx_n) + \lambda d(x_n, x_{n+1}) \leq \\ &\leq \lambda(d(u, x_n) + d(u, x_{n+1}) + d(x_n, x_{n+1})) + \lambda d(Tu, Tx_n). \end{aligned}$$

Consequently

$$d(Tu, Tx_n) \leq \frac{\lambda}{1-\lambda}(d(u, x_n) + d(u, x_{n+1}) + d(x_n, x_{n+1})). \tag{7}$$

From (5) and (7) it follows (6). So we proved that T has at least one fixed-point u in M.

If $v \in M$ is such that $Tv = v$, then by (C) it follows

$$d(u, v) = d(Tu, Tv) \leq q(u, v) \cdot d(u, v)$$

and hence $(1 - q(u, v)) \cdot d(u, v) \leq 0$, which implies that $d(u, v) = 0$, i.e., $v = u$. Thus we proved (i).

Since x was arbitrary, from (5) it follows that (ii) is proved. The relation (iii) follows from (4) when p tends to infinity. So the proof of the theorem is complete.

Observe that from this theorem many corollaries could be derived. For example, if T satisfies the condition

$$d(Tx, Ty) \leq t(d(x, Ty) + d(y, Tx)); \qquad 0 < 2t < 1$$

and M is T-orbitally complete, then, putting $2t = \lambda$, the same assertions as in the theorem 2.5 follow.

2.6. Theorem. *Let T be a mapping of a metric space M into M. If there exists a positive integer k, such that $T^k = TT^{k-1}$ is λ-generalized contraction and M is T-orbitally complete, then*

(i′) *There exists a unique fixed-point $u \in M$ under T,*

(ii′) *$T^n x \to u$ for every $x \in M$ and*

(iii′) $d(T^n x, u) \leq \lambda'^n \cdot \rho(x, Tx)$

where $\lambda' = \lambda^{\frac{1}{k}}$ *and* $\rho(x, Tx) = \max\{\lambda^{-1} d(T^r x, T^{r+k} x) : r = 0, 1, \ldots, k-1\}$.

Proof. It is not hard to see that from the above theorem (i′) and (ii′) follow almost immediately. We shall show only (iii′). Let n be any positive integer. Since T^k is a generalized contraction and $n = mk + r$ with $m = \left[\frac{n}{k}\right]$ and $0 \leq r < k$, from (iii) we have

$$d(T^n x, u) = d(T^{mk} T^r x, u) \leq \lambda^m \cdot d(T^r x, T^k T^r x) =$$

$$= (\lambda^{\frac{1}{k}})^{mk+r-r} \cdot d(T^r x, T^{k+r} x)$$

$$\leq (\lambda^{\frac{1}{k}})^{mk+r-k} \cdot d(T^r x, T^{r+k} x) = (\lambda^{\frac{1}{k}})^n \cdot \lambda^{-1} d(T^r x, T^{r+k} x).$$

Hence

$$d(T^n x, u) \leq (\lambda^{\frac{1}{k}})^n \cdot \max\{\lambda^{-1} d(T^r x, T^{r+k} x) : r = 0, 1, \ldots, k-1\}$$

and the theorem is proved.

Now we shall give a local form of the theorem 2.5.

2.7. Theorem. *Let $x_0 \in M$ and let*

$$B = B(x_0, r) = \{x \in M : d(x_0, x) \leq r\},$$

where M is a metric space. If $T: B \to M$ is a λ-generalized contraction on B, M is T-orbitally complete and

$$d(x_0, Tx_0) \leq (1 - \lambda) \cdot r, \tag{8}$$

then

(i′) *T has in B a unique fixed point u,*

(ii′) *u is a limit point of the sequence*

$$x_0, x_1 = Tx_0, \ldots, x_n = T^n x_0, \ldots$$

and

(iii‴) $d(T^n x_0, u) \leq \lambda^n \cdot r,$

where $\lambda = \sup_{x, y \in B} (q(x, y) + r(x, y) + s(x, y) + 2t(x, y)).$

Proof. Since $\lambda < 1$, from (8) it follows that $x_1 \in B$. By induction we shall show that this sequence is contained in B.

Suppose $x_0, x_1, \ldots, x_m \in B$. Then we may apply (3) for $n = 1, 2, \ldots, m$ and (4) for $n = 0$ and $p = m + 1$. Thus from (4) we have

$$d(x_0, x_{m+1}) \leq \frac{1}{1-\lambda} \cdot d(x_0, Tx_0)$$

and by (8)

$$d(x_0, x_{m+1}) \leq r,$$

i.e., $x_{m+1} \in B$. So the sequence $\{T^n x_0 : n \in N\}$ is contained in B. By same procedures used in the proof of the theorem 2.5, this sequence is Cauchy and has a limit point $u \in M$, which must be fixed under T. Since B is closed, u is in B and so the theorem is proved.

2.8. Definition. *A mapping $T: M \to M$ is said to be an (ε, λ)-uniformly locally generalized contraction if there exists a positive constant ε such that T satisfies (C) whenever $d(x, y) < \varepsilon$.*

2.9. Theorem. *Let T be an (ε, λ)-uniformly locally generalized contraction of a T-orbitally complete metric space M into itself. Then for every $x \in M$ the following alternative holds: either*

(I) *for every integer $s = 0, 1, 2, \ldots,$ one has $d(T^s x, T^{s+1} x) > \varepsilon$, or*

(II) *the sequence $\{T^2 x : n \in N\}$ converges to a fixed point under T.*

Proof. Let $x \in M$ and consider the sequence of numbers $d(x, Tx)$, $d(Tx, T^2 x), \ldots, d(T^s x, T^{s+1} x), \ldots$ There are two possibilities: either

(1) for every integer $s = 0, 1, 2, \ldots,$ one has

$$d(T^s x, T^{s+1} x) > \varepsilon,$$

which is precisely the alternative (I) of the conclusion of the theorem, or else

(2) for some integer $s = s_0$, one has

$$d(T^{s_0} x, T^{s_0+1} x) < \varepsilon.$$

The proof will be complete if we show that (2) implies alternative (II) of the conclusion of the theorem.

Since T is an (ε, λ)-uniformly locally generalized contraction and $d(T^{s_0} x, T^{s_0+1} x) < \varepsilon$, then repeating the procedure used in theorem 2.5. we obtain that (3) holds for $T^{s_0+1} x$ and $T^{s_0+2} x$, i.e.

$$d(T^{s_0+1} x, T^{s_0+2} x) \leq \lambda d(T^{s_0} x, T^{s_0+1} x) < \lambda \varepsilon < \varepsilon,$$

where $\lambda = \sup\{q(x,y)+r(x,y)+s(x,y)+2t(x,y)\}$, $x, y \in M$ and $d(x,y)<\varepsilon$. By induction it follows that

$$d(T^{s_0+p}x,\ T^{s_0+p+1}x) \le \lambda^p \cdot d(T^{s_0}x,\ T^{s_0+1}x) < \varepsilon$$

for every integer $p=0, 1, 2, \ldots$, and hence, as in the theorem 2.5., for $n>s_0$ we have

$$d(T^n x,\ T^{n+p}x) \le \frac{\lambda^{n-s_0}}{1-\lambda}\, d(T^{s_0}x,\ T^{s_0+1}x). \tag{9}$$

Thus the sequence

$$x,\ Tx,\ T^2x,\ \ldots,\ T^n x,\ \ldots$$

is a Cauchy sequence and let $u \in M$ be such that

$$u = \lim_{n\to\infty} T^n x = \lim_{p\to\infty} T^{s+p}x.$$

Since $\lambda<1$ and $d(T^n x, u) \le \dfrac{\lambda^{n-s_0}}{1-\lambda} d(T^{s_0}x, T^{s_0+1}x)$ for $n>s_0$, it follows that there is an integer $n_0>s_0$ such that

$$d(T^n x, u) < \varepsilon$$

for every $n>n_0$. Then we obtain that (7) holds for all $n>n_0>s_0$, i.e.

$$d(Tu,\ TT^n x) \le \frac{\lambda}{1-\lambda}\,(d(u,\ T^n x)+d(u,\ T^{n+1}x)+d(T^n x,\ T^{n+1}x)).$$

Hence, letting n tend to infinity, it follows $d(Tu, u)=0$, i.e. $Tu=u$, which concludes the proof.

Observe that if $v \in M$ is a fixed point under T and $v \ne u$, then must be $d(u, v) > \varepsilon$. Indeed, if $0<d(u,v)<\varepsilon$ and u and v are fixed points, it follows

$$d(u,v) = d(Tu,\ Tv) \leqslant q(u,v)\,d(u,v) + t(u,v)\,(d(u,\ Tv)+d(v,\ Tu)) =$$
$$= (q(u,v)+2t(u,v))\,d(u,v) \le \lambda d(u,v) < d(u,v),$$

which desired contradiction.

As a special case of the above theorem we have the following result:

2.10. Corollary. Let $T\colon M \to M$ be an (ε, λ)-uniformly locally generalized contraction on a T-orbitally complete metric space M into itself. If for every $x \in M$ there exists an integer $n(x)$ such that

$$d(T^{n(x)}x,\ T^{n(x)+1}x) < \varepsilon,$$

and if u and v two fixed points under T then $d(u,v)<\varepsilon$, then T has exactly one fixed point, and all sequences $\{T^n x : n \in N\}$, $x \in M$, converge to the unique fixed point of T.

3. Generalized contractions need not be continuous, as it was shown by the example 1, but are such that fixed point theorems may be proved without assumption of an orbitally continuity. The following theorem shows that every generalized contraction is orbitally continuous in the sense of the definition 2.3.

3.1. Theorem. *Let $T: M \to M$ be a λ-generalized contraction mapping of a metric space M into itself. If $u \in M$ is such that $u = \lim_{i\to\infty} T^{n_i} x$ for some $x \in M$, then $Tu = \lim_{i\to\infty} T^{n_i+1} x$.*

Proof. Let u and x in M be such that $u = \lim_{i\to\infty} T^{n_i} x$. Consider the sequence $\{T^n x : n \in N\}$ which contains the sequence $\{T^{n_i} x : i \in N\}$ as a subsequence. Since T is a generalized contraction, it follows that the sequence $\{T^n x : n \in N\}$ must be Cauchy, as it was shown in the first part of the proof of the theorem 2.5. Since $\{T^n x : n \in N\}$ is a Cauchy sequence and contains a subsequence $\{T^{n_i} x : i \in N\}$ such that $\lim_{i\to\infty} T^{n_i} x = u$, it follows that $\lim_{n\to\infty} T^n x = u$. Then, from (7) one has $\lim_{n\to\infty} d(Tu, TT^n x) = 0$, i.e. $Tu = \lim_{n\to\infty} T^{n+1} x$, which implies $Tu = \lim_{i\to\infty} T^{n_i+1} x$. This completes the proof.

Now we shall give an example which shows that a contraction type mapping need not be orbitally continuous on a compact metric space.

Example 3. Let $M = [0, 1] \subset R^1$ and $T: M \to M$ be such mapping that $Tx = \frac{x}{2}$ for $x \neq 0$ and $T0 = 1$. Then T is a contraction type mapping with $q(x, y) = \frac{1}{2}$ and $\delta(x, y) = 2$ for all $x, y \in M$. But T is not T-orbitally continuous. Indeed, $0 \in M$ is such that $0 = \lim_{n\to\infty} T^n x$ for any $x \in M$, but $1 = T0 \neq \lim_{n\to\infty} TT^n x = 0$.

REFERENCES

[1] Banach S., *Sur les opérations dans les ensembles abstraits et leur applications aux équations intégrales*, Fund. Math. 3 (1922), p. 160.

[2] Ćirić Lj., *On contraction type mappings*, Math. Balcanica 1 (1971), 52—57.

[3] Kannan R., *On certain sets and fixed-point theorems*, Roum. Math. pure appl. 14 (1969), 51—54.

[4] Kannan R., *Some results on fixed points — II*, Amer. Math. Monthly, 76 (1969), p. 406.