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ORLICZ SPACES CONNECTED WITH STRONG SUMMABILITY. III
ON STRONG ($\mathfrak{M}$, φ) — SUMMABILITY

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1. Modular spaces of strongly summable sequences were investigated in papers of J. Musielak and W. Orlicz [11], J. Musielak [9], and W. Orlicz [16]. These investigations were limited essentially to the case of strong summability to zero by means of the method of first arithmetic means. Generalizing the definition of strong summability of J. Musielak and W. Orlicz, I carried out in papers [20] and [21] some investigations under more general assumption on the methods of strong summability, defining and studying Orlicz spaces of sequences strongly (A, φ)-summable to zero. Expanding these considerations and applying important and numerous suggestions of Professor W. Orlicz and valuable remarks of Docent J. Musielak, I extended in [22] the problematic to the case of summability to zero of functions. The problem of strong summability of functions differs essentially from that of sequences, although the ideas of some proofs and also the results are in many cases analogous, comparing papers [20] and [22]. Namely, in connection with strong summability of functions there arises a number of new technical problems in proofs, here, quite important are the assumptions on the kernel of the integral transformation.

In the present paper, I am replacing particular measures by a general family $\mathfrak{M}$ of nonnegative measures, when defining the strong ($\mathfrak{M}$, φ)-summability. The assumptions on the family $\mathfrak{M}$ are enough general in order to cover the special cases of purely atomic measures or atomless measures considered in [20] and [22]. In this sense, the results which follow can be considered as generalizations of those of J. Musielak and W. Orlicz, and also of those obtained in [20] and [22].

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1.1. We shall apply the following notation. Let T be a locally compact Hausdorff topological space, and let $\tau_0 \notin \mathrm{T}$. Let us denote $\mathrm{T}_0 = \mathrm{T} \cup \{\tau_0\}$. Compact sets in T will be denoted by symbols $Z, Z_1, Z_2, \ldots$. By a neighbourhood U of the point τ_0 in T_0 we shall mean the complement in T_0 of any compact set in T, i.e. $U = Z'$. In the following we shall write $\tau_0 = \infty$.

Now, by $\mathfrak{S}$ we shall denote a σ-algebra of subsets of an abstract set E and we take a fixed σ-ring $\mathfrak{S}_0 \subset \mathfrak{S}$. Sets from $\mathfrak{S}_0$ will be denoted by $\Omega, K, K_1, K_2, \ldots$

1.2. In the sequel $\mathfrak{M} = \{\mu_\tau\}$, $\tau \in \mathrm{T}$, will denote a family of nonnegative measures, defined on the σ-algebra $\mathfrak{S}$ of subsets the set E. Sets belonging to $\mathfrak{S}$ and functions on E measurable with respect to $\mathfrak{S}$, will be called $\mathfrak{M}$-measurable.

A set B will be called an $\mathfrak{M}$-zero set, if and only if, $\mu_\tau B = 0$ for every $\tau \in T$. Let $\mathfrak{N}_\tau$ be the σ-ring of sets in E of μ_τ-measure zero, then $\mathfrak{N} = \bigcap_{\tau \in T} \mathfrak{N}_\tau$ is the σ-ring of $\mathfrak{M}$-zero sets.

1.3. The following notations will be used for spaces of functions defined on E. By $\mathfrak{X}$ we shall denote the space whose elements are classes of $\mathfrak{M}$-measurable real-valued functions, where two functions belong to the same class if and only if they differ on a set belonging to the σ-ring $\mathfrak{N}$. We shall say briefly, that $\mathfrak{X}$ is the real space of all $\mathfrak{M}$-measurable elements defined on E. $\mathfrak{X}_\Omega$ will denote the space of such elements belonging to the space $\mathfrak{X}$ whose support $\Omega \in \mathfrak{S}_0$, and $\mathfrak{X}_b$ will mean the spaces of all bounded functions in $\mathfrak{X}$. Moreover, $\mathfrak{X}_\Omega^s$ will denote the space of simple functions belonging to $\mathfrak{X}_\Omega$.

Elements of $\mathfrak{X}$ will be denoted by $x(t)$, $y(t)$, $z(t)$, ... Often we shall write $x, y, z, \ldots$ in place of $x(t)$, $y(t)$, $z(t)$, ..., $\sup x$ in place of $\sup x(t)$. Pointwise convergence and uniform convergence in E of $x_n(t)$ to $x(t)$ as $n \to \infty$ will be denoted by $x_n \to x$ and $x_n \rightrightarrows x$, respectively. If $x, y \in \mathfrak{X}$, the symbol $x \leqslant y$ will mean that $x(t) \leqslant y(t)$ with the exception of a set belonging to the σ-ring $\mathfrak{N}$. The relation $\leqslant$ defines a partial order in $\mathfrak{X}$, and $\mathfrak{X}$ is a linear lattice with respect to this relation, which is order-complete, i.e. each non-empty set of elements of $\mathfrak{X}$ which is bounded from above possesses a last upper bound belonging to $\mathfrak{X}$. The supremum or infimum of two elements $x, y \in \mathfrak{X}$ will be denoted by $x \vee y$ or $x \wedge y$, respectively; the symbols $x_1 \vee x_2 \vee \cdots \vee x_n$, $V x_n$, etc., have analogous meaning.

1.4. By a φ-function we understand (as in paper [23]) a continuous, non-decreasing function $\varphi(u)$ defined for $u \geqslant 0$ and such that $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$ and $\varphi(u) \to \infty$ as $u \to \infty$. φ-functions will be denoted by letters $\varphi, \psi, \ldots$, and their inverses, by $\varphi_{-1}, \psi_{-1}, \ldots$

A φ-function φ is said to satisfy the condition (Δ_2) for large u, if for some constants $k > 1$, $u_0 > 0$ there is satisfied the inequality

$$(\Delta_2) \qquad \varphi(2u) \leqslant k\varphi(u) \quad \text{for} \quad u \geqslant u_0.$$

A function φ is called nonweaker than ψ for large u, if for some constants $c, b, l, k, u_0 > 0$ there is satisfied the inequality $c\psi(lu) \leqslant b\varphi(ku)$ for $u \geqslant u_0$, we denote this relation writing $\psi \overset{l}{\prec} \varphi$.

A function ψ is called equivalent to φ for large u, if there are constants $a, b, c, k_1, k_2, l, u_0 > 0$ such that $a\varphi(k_1 u) \leqslant c\psi(lu) \leqslant b\varphi(k_2 u)$ for $u \geqslant u_0$; we denote it writing $\psi \overset{l}{\sim} \varphi$. Obviously, $\varphi \overset{l}{\sim} \psi$ if and only if $\psi \overset{l}{\prec} \varphi$ and $\varphi \overset{l}{\prec} \psi$, simultaneously.

A convex φ-function φ is said to satisfy the conditions (0_1) or (∞_1), if

$$(0_1) \quad \frac{\varphi(u)}{u} \to 0 \text{ as } u \to 0+, \qquad (\infty_1) \quad \frac{\varphi(u)}{u} \to \infty \text{ as } u \to \infty,$$

respectively.

The function φ^* conjugate to a convex φ-function satisfying the conditions (0_1) and (∞_1), is defined by the equality $\varphi^*(v) = \sup_{v \geqslant 0} (uv - \varphi(u))$. The function φ^* is again a convex φ-function satisfying the conditions (0_1) and (∞_1). Moreover, it satisfies the following Young inequality $uv \leqslant \varphi(u) + \varphi^*(v)$, (see e.g. [4], [5], [6], [13] and also [23]).

The symbol $\varphi(|x|)$ means in the following the function $\varphi(|x(t)|)$ defined for $t \in E$.

1.5. Let a family $\mathfrak{M}=\{\mu_\tau\}$ of measures and a φ-function φ be given. We denote for any $x\in\mathfrak{X}$

$$\sigma_\varphi(\tau, x)=\int_E \varphi(|x|)\,d\mu_\tau.$$

We say that the integral transformation $\sigma_\varphi(\tau, x)$ tends to zero as $\tau\to\infty$ and we write $\sigma_\varphi(\tau, x)\to 0$ as $\tau\to\infty$, if for any number $\varepsilon>0$ there exists a set Z compact in T such that $\tau\notin Z$ implies $\sigma_\varphi(\tau, x)<\varepsilon$. We introduce now the following notation:

$$\mathfrak{X}_{\varphi_0}=\{x\in\mathfrak{X}:\sigma_\varphi(\tau, x)<\infty \text{ for } \tau\in \mathrm{T};\ \sigma_\varphi(\tau, x)\to 0 \text{ as } \tau\to\infty\},$$

$$\mathfrak{X}_\varphi=\{x\in\mathfrak{X}:\lambda x\in\mathfrak{X}_{\varphi_0} \text{ for arbitrary } \lambda>0\},$$

$$\mathfrak{X}^*_\varphi=\{x\in\mathfrak{X}:\lambda x\in\mathfrak{X}_{\varphi_0} \text{ for some } \lambda>0\}.$$

Since the family of measures is fixed in our further considerations, we do not point out the dependence of the above spaces on this family.

The elements of the space χ^*_φ are called strongly $(\mathfrak{M}, \varphi)$-summable to zero.

1.6. In order to derive a theory of these spaces, the following special assumptions on the family of measures $\mathfrak{M}$ will be used in some of our further considerations:

1° For every set $K\in\mathfrak{G}_0$, $\sigma_\varphi(\tau, x\chi_K)$ is a continuous function of the variable $\tau\in\mathrm{T}$, where χ_K means the characteristic function of the set K.

2° For every $\mathfrak{M}$-measurable function x for which $\sigma_\varphi(\tau, x)$ is finite for all $\tau\in\mathrm{T}$ and $\sigma_\varphi(\tau, x)\to 0$ as $\tau\to\infty$, the integral remainders are uniformly small on set Z compact in T, i.e. for each $\varepsilon>0$ there exists a set $K\in\mathfrak{G}_0$ and a set Z compact in T, there holds $\int_{K'}\varphi(|x|)\,d\mu_\tau<\varepsilon$ for all $\tau\in Z$.

3° For an arbitrary set $K\in\mathfrak{G}_0$ and for any $\varepsilon>0$ there exists a set Z compact in T such that $\tau\notin Z$ implies $\mu_\tau K<\varepsilon$; to be brief we shall denote it writing $\mu_\tau K\to 0$ as $\tau\to\infty$.

4° The family of measures $\mathfrak{M}$ is uniformly bounded, i.e. there exists a constant $C>0$ such that $\mu_\tau E\leqslant C$ for all $\tau\in\mathrm{T}$.

5° Let $K\in\mathfrak{G}_0$, and let us denote $A(K)=\sup_{\tau\in\mathrm{T}}\mu_\tau K$.

(a) For an arbitrary set $K\in\mathfrak{G}_0$ there exists a set Z compact in T such that $A(K)=\sup_{\tau\in Z}\mu_\tau K$.

(b) There exist constants δ, c $(\delta>0,\ 0<c\leqslant 1)$ such that for every number η satisfying the inequalities $0<\eta\leqslant\delta$ there exists a set $K\in\mathfrak{G}_0$ such that $c\eta\leqslant A(K)\leqslant\eta$.

2.1. The characteristic function χ_B of a set $B\in\mathfrak{G}$ belongs to $\mathfrak{X}^*_\varphi$, if and only if, $\mu_\tau B<\infty$ for $\tau\in\mathrm{T}$ and $\mu_\tau B\to 0$ as $\tau\to\infty$. In particular, conditions 3° and 4° are sufficient in order that characteristic functions of all sets from $\mathfrak{G}_0$ belong to χ^*_φ.

It is easily verified that the inclusion $\mathfrak{X}^s_\Omega\subset\chi^*_\varphi$ holds, if and only if, $\mu_\tau\Omega<\infty$ for $\tau\in\mathrm{T}$ and $\mu_\tau\Omega\to 0$ as $\tau\to\infty$. Moreover, the same two conditions are necessary and sufficient in order that $\mathfrak{X}_b\cap\mathfrak{X}_\Omega\subset\mathfrak{X}^*_\varphi$. In particular, the conditions 3° and 4° on the family of measures $\mathfrak{M}$ are sufficient in order that all simple functions constant on sets from $\mathfrak{G}_0$ belong to χ^*_φ.

2.2. If the family of measures $\mathfrak{M}$ possesses the properties 1° and 2° then $\sigma_\varphi(\tau, x)$ is a continuous function of the variable τ for all $\tau \in T$.

To prove this statement, let us take any $\varepsilon > 0$, and let $K \in \mathfrak{S}_0$. According to 1°, $\sigma_\varphi(\tau, x\chi_K)$ is a continuous function of the variable τ. Hence $|\sigma_\varphi(\tau, x\chi_K) - \sigma_\varphi(\tau_0, x\chi_K)| < \frac{\varepsilon}{2}$ for τ sufficiently near to τ_0. On the other hand, by 2°, for each $\varepsilon > 0$ there exist a set $K \in \mathfrak{S}_0$ and a set Z compact in T such that the inequality

$$\left| \int_{K'} \varphi(|x|)\, d(\mu_\tau - \mu_{\tau_0}) \right| < \frac{\varepsilon}{2}$$

holds for $\tau, \tau_0 \in Z$. Thus we obtain finally

$$|\sigma_\varphi(\tau, x) - \sigma_\varphi(\tau_0, x)| \leqslant |\sigma_\varphi(\tau, x\chi_K) - \sigma_\varphi(\tau_0, x\chi_K)| + \left| \int_{K'} \varphi(|x|)\, d(\mu_\tau - \mu_{\tau_0}) \right| < \varepsilon.$$

2.3. $\mathfrak{X}_\varphi^*$ is a linear space and $\mathfrak{X}_{\varphi_0}$ is a convex set in χ_φ^* (in particular if $\lambda_0 x \in \mathfrak{X}_{\varphi_0}$ for some $\lambda_0 > 0$, then $\lambda x \in \mathfrak{X}_{\varphi_0}$ for all $0 \leqslant \lambda \leqslant \lambda_0$). With the order relation defined in $\mathfrak{X}$, $\mathfrak{X}_\varphi^*$ is a linear lattice and a sublattice of $\mathfrak{X}$. The lattice $\mathfrak{X}_\varphi^*$ is σ-order complete, and even order-complete. The proof runs the same lines as in [20] and [22].

2.4. If the family of measures $\mathfrak{M}$ possesses the property 4°, then there holds $\mathfrak{X}_\varphi^* \cap \mathfrak{X}_b = \mathfrak{X}_\psi^* \cap \mathfrak{X}_b$ for any two φ-functions φ and ψ.

If the family of measures $\mathfrak{M}$ possesses the property 4°, then there holds $\mathfrak{X}_\varphi^* \cap \mathfrak{X}_b = \mathfrak{X}_\varphi \cap \mathfrak{X}_b$ for arbitrary φ-function φ.

The proofs of these facts are performed in the same manner as in [20], applying the inequality (+) from [16], p. 333.

2.5. Let the family of measures $\mathfrak{M}$ possess the property 4°. If $\psi \overset{l}{\prec} \varphi$, then $\mathfrak{X}_\varphi^* \subset \mathfrak{X}_\psi^*$ and $\mathfrak{X}_\varphi \subset \mathfrak{X}_\psi$.

To prove this statement, let us take $x \in \mathfrak{X}_\varphi^*$, i.e. $\lambda_0 x \in \mathfrak{X}_{\varphi_0}$ for some $\lambda_0 > 0$. Let us denote $E_1 = \{t \in E : \lambda_0 k^{-1} |x(t)| \geqslant u_0\}$, where k and u_0 are taken from the definition of the relation $\overset{l}{\prec}$. Then $\psi(\lambda_0 k^{-1}|x|) \leqslant b\varphi(\lambda_0 |x|)$ in the set E_1. We define an element x_1 as follows: $x_1 = x$ in $E \setminus E_1$, $x_1 = 0$ in E_1. Obviously, $x_1 \in \mathfrak{X}_\varphi^*$ and, by 2.4, $x_1 \in \mathfrak{X}_\varphi^* \cap \mathfrak{X}_b = \mathfrak{X}_\psi^* \cap \mathfrak{X}_b$, i.e. $x_1 \in \mathfrak{X}_\psi^*$. But $x_2 = x - x_1 \in \mathfrak{X}_\psi^*$, because

$$\int_E \psi(\lambda_0 k^{-1}|x|)\, d\mu_\tau \leqslant b \int_E \varphi(\lambda_0 |x|)\, d\mu_\tau.$$

Finally $x = x_1 + x_2 \in \mathfrak{X}_\psi^*$. The inclusion $\mathfrak{X}_\varphi \subset \mathfrak{X}_\psi$ is proved in the same manner.

3.1. In the sequel we shall suppose always that the family of measures $\mathfrak{M}$ possesses the properties 1° and 2°. By 2.2, $\sigma_\varphi(\tau, x)$ is a continuous function of τ for $\tau \in T$. From the condition $\sigma_\varphi(\tau, x) \to 0$ as $\tau \to \infty$ follows that $\sup_{\tau \in T} \sigma_\varphi(\tau, x) < \infty$ for $x \in \mathfrak{X}_{\varphi_0}$.

We define in $\mathfrak{X}_\varphi^*$ the following functional:

$$\rho_\varphi(x) = \begin{cases} \sup_{\tau \in T} \int_E \varphi(|x|)\, d\mu_\tau & \text{for } x \in \mathfrak{X}_{\varphi_0}, \\ \infty & \text{for } x \in \chi_\varphi^* \setminus \mathfrak{X}_{\varphi_0}. \end{cases}$$

By assumptions 1° and 2° on the family of measures $\mathfrak{M}$, the functional $\rho_\varphi(x)$ is a modular in $\mathfrak{X}_\varphi^*$ in the sense of the definition assumed by Orlicz in [15], [17], i.e. it satisfies the following conditions

A. $\rho_\varphi(x)=0$ if and only if $x=0$,

B. $\rho_\varphi(x_1)\leqslant\rho_\varphi(x_2)$ if $|x_1|\leqslant|x_2|$,

C. $\rho_\varphi(x_1\vee x_2)\leqslant\rho_\varphi(x_1)+\rho_\varphi(x_2)$ if $x_1\geqslant 0,\ x_2\geqslant 0$,

D. $\rho_\varphi(\lambda x)\to 0$ if $\lambda\to 0+$.

Simple proofs of these properties will be omitted (see [22]).

Let us remark that condition C implies the inequality $\rho_\varphi(\alpha x_1+\beta x_2)\leqslant\leqslant\rho_\varphi(x_1)+\rho_\varphi(x_2)$ for $\alpha,\beta\geqslant 0$, $\alpha+\beta=1$; moreover, in case when φ is an s-convex φ-function, the modular may be defined assuming the conditions A, B and replacing C by

C_s. $\rho_\varphi(\alpha x_1+\beta x_2)\leqslant\alpha^s\rho_\varphi(x_1)+\beta^s\rho_\varphi(x_2)$, where $\alpha,\beta\geqslant 0;\alpha^s+\beta^s=1;\ 0<s\leqslant 1$.

3.2. From 3.1 follows that $\mathfrak{X}_\varphi^*$ is a modular space, partially ordered in the sense of the definition in [17]. Hence a nonnegative, finite-valued functional $\|\cdot\|_\varphi$ may by defined in $\mathfrak{X}_\varphi^*$ by means of the formula

$$(*)\qquad \|x\|_\varphi=\inf\left\{\varepsilon>0:\rho_\varphi\left(\frac{x}{\varepsilon}\right)\leqslant\varepsilon\right\}.$$

It is easily verified that the functional (*) is an F-norm (see [10], [15], [17]). If φ is an s-convex φ-function, then an s-homogeneous norm may be defined in $\mathfrak{X}_\varphi^*$ by means of the formula

$$(**)\qquad \|x\|_{s\varphi}=\inf\left\{\varepsilon>0:\rho_\varphi\left(\frac{x}{\varepsilon^{1/s}}\right)\leqslant 1\right\},$$

(see [6], [11]). It is known ([7], [11], [14]), that both norms (**) and (*) are equivalent. Let us remark that 3.1. B implies monotonicity of the norm $\|\cdot\|_{s\varphi}$, immediately. If φ is a convex φ-function, then besides the norm (**) one can define in $\mathfrak{X}_\varphi^*$ a norm by means of the modular

$$\overset{0}{\rho}_\varphi(x)=\sup_{\tau\in T}\int_E\varphi(|x|)\,d\mu_\tau \text{ for } x\in\mathfrak{X}_\varphi^*.$$

It is easily verified that $\overset{0}{\rho}_\varphi(x)$ satisfies the conditions 3.1. A, B, D and C_1; hence we can define a norm

$$\|x\|_\varphi^0=\inf\left\{\varepsilon>0:\overset{0}{\rho}_\varphi\left(\frac{x}{\varepsilon}\right)\leqslant 1\right\}.$$

Let us remark that $\|x\|_\varphi^0\leq\|x\|_{1\varphi}$; moreover, $\|x\|_\varphi^0=\|x\|_{1\varphi}$ for $x\in\mathfrak{X}_\varphi$, since for such x there holds $\overset{0}{\rho}_\varphi(x)=\rho_\varphi(x)$.

3.3. The space χ_φ is a linear subspace of the space $\mathfrak{X}_\varphi^*$, closed with respect to the norm (*) (see the proof in [20], and also in [22]).

In our further considerations, under $\mathfrak{X}_\varphi$, $\mathfrak{X}_\varphi^*$, ... we shall understand always the respective spaces of elements provided with norm (*) or any norm equivalent to (*).

3.4. Let us suppose that the family of measures $\mathfrak{M}$ possesses properties 3° and 4°, and let us denote $A(\Omega)=\sup\limits_{\tau\in T}\mu_\tau\Omega$. Let φ be an s-convex φ-function.

Then φ is strictly increasing, and the s-homogeneous norm of the characteristic function of a set $\Omega \in \mathfrak{S}_0$ is easily determined. Obviously, $\chi_\Omega \in \mathfrak{X}_\varphi^*$, and

$$\|\chi_\Omega\|_{s\varphi} = \left(\varphi_{-1}\left(\frac{1}{A(\Omega)}\right)\right)^{-s}.$$

3.5. Let us yet note that if φ is a convex φ-function satisfying the conditions (0_1) and (∞_1), and if φ^* is complementary to φ, and the family of measures $\mathfrak{M}$ possesses the property 3° (besides 1° and 2°), then the following norm may be introduced in $\mathfrak{X}_\varphi^*$:

$$\|x\|_\varphi^* = \sup_y \sup_{\tau \in T} \left| \int_E xy\, d\mu_\tau \right|,$$

where the supremum is taken over the set of all $y \in \mathfrak{X}_{\varphi^*}$ satisfying the inequality $\rho_{\varphi^*}(y) \leqslant 1$. It is easily verified that $\|\cdot\|_\varphi^*$ is a monotone B-norm in $\mathfrak{X}_\varphi^*$.

4.1. The space χ_φ^* is complete with respect to the norm $\|\cdot\|_\varphi$.

In the proof we shall apply the following theorem from [17]: If axiom C. II is satisfied in a modular space $X(\rho)$, then this space is complete with respect to any norm generated by ρ. Hence, it is sufficient to show that C. II holds in $\mathfrak{X}_\varphi^*$, i.e. that the conditions $x_n \in \mathfrak{X}_\varphi^*$, $x_n \geqslant 0$ for $n=1, 2, \ldots$, $\rho_\varphi(x_1) + \rho_\varphi(x_2) + \cdots < \infty$ imply existence of $x_0 = Vx_n \in \mathfrak{X}_\varphi^*$.

First let us remark that we may limit ourselves in the proof to nonnegative elements. Let $\{z_n\}$ be a sequence of elements of a partially ordered space, and let $z_n \leqslant z$, $n=1, 2, \ldots$ We consider the sequence $x_1 = z_1$, $x_2 = z_1 \vee z_2, \ldots, x_n = z_1 \vee z_2 \vee \cdots \vee z_n, \ldots$ It is seen that $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n \leqslant \cdots$ and $x_n \leqslant z$ for $n=1, 2, \ldots$ Hence $0 \leqslant x_2 - x_1 \leqslant x_3 - x_1 \leqslant \cdots \leqslant x_n - x_1 \leqslant \cdots$; thus the sequence $\{x_n - x_1\}$ is nonnegative and $0 \leqslant x_n - x_1 \leqslant z - x_1$ for $n=1, 2, \ldots$ If $V(x_n - x_1) = u$ exists, then $Vx_n = u + x_1$ also exists. We show yet that the least upper bounds of the sequences $\{x_n\}$ and $\{z_n\}$ are the same. We know that $z_n \leqslant x_n = z_1 \vee z_2 \vee \cdots \vee z_n$. Let $z_n \leqslant z$ for $n=1, 2, \ldots$, then also $x_n \leqslant z$ for $n=1, 2, \ldots$, i.e. $Vx_n \leqslant z$. On the other hand, $Vx_n \geqslant x_k \geqslant z_k$ for $k=1, 2, \ldots$, i.e. Vx_n is the least upper bound of the sequence $\{z_n\}$; thus Vz_n exists and is equal to Vx_n.

Now, let $x_n \in \mathfrak{X}_\varphi^*$, $x_n \geqslant 0$ for $n=1, 2, \ldots$ We consider a sequence $y_k = x_1 \vee x_2 \vee \cdots \vee x_k$. Since $\varphi(y_k) \leqslant \varphi(x_1) + \varphi(x_2) + \cdots + \varphi(x_k)$, we get

$$\int_E \varphi(y_k)\, d\mu_\tau \leqslant \int_E \varphi(x_1)\, d\mu_\tau + \cdots + \int_E \varphi(x_k)\, d\mu_\tau$$

and we obtain for a given

$$(+) \qquad \sigma_\varphi(\tau, y_k) \leqslant \sum_{j=1}^{k} \sigma_\varphi(\tau, x_j) \leqslant \sum_{j=1}^{\infty} \sigma_\varphi(\tau, x_j) \leqslant \sum_{j=1}^{\infty} \rho_\varphi(x_j) < \infty.$$

But y_k is an increasing sequence, hence we have $y_k(t) \to x_0(t)$ for all t, where $x_0 = Vx_k$. We show that $x_0 \in \mathfrak{X}_\varphi^*$. From the above it follows that $\lim_{k\to\infty} \sigma_\varphi(\tau, y_k) = \sigma_\varphi(\tau, x_0) < \infty$ for each $\tau \in T$. By $(+)$, we have

$$\sigma_\varphi(\tau, x_0) \leqslant \sum_{j=1}^{\infty} \sigma_\varphi(\tau, x_j).$$

Hence there holds for each $\tau \in T$

$$\sigma_\varphi(\tau, x_0) \leqslant \sigma_\varphi(\tau, x_1) + \cdots + \sigma_\varphi(\tau, x_{k-1}) + \rho_\varphi(x_k) + \rho_\varphi(x_{k+1}) + \cdots$$

By the assumption, given $\varepsilon > 0$ we have

$$\sum_{j=k}^{\infty} \rho_\varphi(x_j) < \varepsilon$$

for sufficiently large k, i.e. $\sigma_\varphi(\tau, x_0) \leqslant \sigma_\varphi(\tau, x_1) + \cdots + \sigma_\varphi(\tau, x_{k-1}) + \varepsilon$ for each $\tau \in T$ and for sufficiently large k. Since $\sigma_\varphi(\tau, x_j) \to 0$ as $\tau \to \infty$, for any $\varepsilon > 0$ there exists a set Z compact in T such that

$$\sigma_\varphi(\tau, x_1) + \sigma_\varphi(\tau, x_2) + \cdots + \sigma_\varphi(\tau, x_{k-1}) < \varepsilon$$

for all $\tau \notin Z$. Thus we obtained finally that $\sigma_\varphi(\tau, x_0) < 2\varepsilon$ for every $\tau \notin Z$, i.e. $\sigma_\varphi(\tau, x_0) \to 0$ as $\tau \to \infty$.

Let us yet remark that since the space $\mathfrak{X}_\varphi$ is closed with respect to the norm $\|\cdot\|_\varphi$ (see 3.3), it is complete with respect to the same norm.

From the above proof it follows also that $\rho_\varphi(x_0) \leqslant \rho_\varphi(x_1) + \rho_\varphi(x_2) + \cdots$

4.2.1. We shall say that the family of measures $\mathfrak{M} = \{\mu_\tau\}$, $\tau \in T$, is separable, if it satisfies the following condition: there exists a sequence of sets $K_n \in \mathfrak{S}_0$, $n = 1, 2, \ldots$, such that for an arbitrary set B belonging to the σ-algebra $\mathfrak{S}$ there exists a subsequence K_{ni} for which $\mu_\tau(K_{ni} \dot{-} B) \to 0$ uniformly in T.

4.2.2. If the family of measures $\mathfrak{M}$ is separable and possesses the properties 3° and 4°, then the space $\mathfrak{X}_\varphi$ is separable with respect to the norm $\|\cdot\|_\varphi$.

To prove this theorem let us first remark that there exists a sequence of sets $K_n \in \mathfrak{S}_0$ such that for every $B \in \mathfrak{S}$ one can extract a subsequence K_{ni} for which $\rho_\varphi(\lambda(\chi_{K_{n_i}} - \chi_B)) \to 0$ for an arbitrary $\lambda > 0$. Indeed, we have

$$\rho_\varphi(\lambda(\chi_{K_{n_i}} - \chi_B)) = \sup_{\tau \in T} \int_E \varphi(\lambda |\chi_{K_{n_i}} - \chi_B| \, d\mu_\tau = \varphi(\lambda) \sup_{\tau \in T} \int_{K_{n_i} \dot{-} B} d\mu_\tau =$$
$$= \varphi(\lambda) \sup_{\tau \in T} \mu_\tau(K_{ni} \dot{-} B);$$

but $\mu_\tau(K_{ni} \dot{-} B) \to 0$ uniformly in T, by definition. Hence $\rho_\varphi(\lambda(\chi_{K_{n_i}} - \chi_B)) \to 0$.

We shall show that the set of simple functions of the form

$$x(t) = \sum_{i=1}^{n} w_i \chi_{K_i}(t),$$

where w_i are rational numbers, is a countable, dense set in $\mathfrak{X}_\varphi$. First, we choose an arbitrary simple function $y(t) = \sum_{j=1}^{m} c_j \chi_{B_j}(t)$, where $B_j \in \mathfrak{S}$ and c_j are real numbers. Then it is easily seen that

$$\rho_\varphi(\lambda(x - y)) = \sup_{\tau \in T} \sum_i \sum_j \varphi(\lambda |w_i - c_j|) \mu_\tau(K_i \cap B_j) \leqslant$$
$$\leqslant \sup_{\tau \in T} \sum_i \varphi(\lambda |w_i - c_i|) \mu_\tau(K_i \cap B_i) + \sup_{\tau \in T} \sum_{i \neq j} \varphi(\lambda |w_i - c_j|) \mu_\tau(K_i \cap B_j).$$

Obviously, the difference $|w_i - c_i|$ may be arbitrarily small. Moreover, because $\mu_\tau(K_i \cap B_j) \leqslant \mu_\tau(K_i \dot{-} B_i) \leqslant \sup_{\tau \in T} \mu_\tau(K_i \dot{-} B_i)$, it follows form the assumption that the sets K_i may be chosen in such a manner that $\mu_\tau(K_i \cap B_j)$ are also arbitrarily small for every $\tau \in T$. Finally, we obtain that for an arbitrary $\varepsilon > 0$ there exists an x such that $\rho_\varphi(\lambda(x-y)) < \varepsilon$, where $\lambda > 0$.

It is yet to be proved that the set of simple functions is dense in $\mathfrak{X}_\varphi$. First we show that if $x \in \mathfrak{X}_\varphi$ and $B \in \mathfrak{S}$, then the integrals $\int_B \varphi(|x|)\, d\mu_\tau$ are uniformly absolutely continuous with respect to τ, i.e. the following condition is satisfied: for every $\varepsilon > 0$ there exists a number $\delta > 0$ such that for every set $B \in \mathfrak{S}$, the inequality $\sup_{\tau \in T} \mu_\tau B < \delta$ implies the inequality $\sup_{\tau \in T} \int_B \varphi(|x|)\, d\mu_\tau < \varepsilon$. This is obtained applying absolute continuity of the integral for each τ separately and the definition of supremum, because given any $\varepsilon > 0$ there is a $\tau_\varepsilon \in T$ for which

$$\sup_{\tau \in T} \int_B \varphi(\lambda |x|)\, d\mu_\tau < \int_B \varphi(\lambda |x|)\, d\mu_{\tau_\varepsilon} + \varepsilon,$$

and the first term at the right-hand side of this inequality is less than ε for $\sup_{\tau \in T} \mu_\tau B < \delta$, where $\delta > 0$ depends on ε.

Now, let us suppose that $x \in \mathfrak{X}_\varphi$, $x \geqslant 0$, and let us define

$$x_n(t) = \begin{cases} \dfrac{k-1}{2^n} & \text{for } \dfrac{k-1}{2^n} \leqslant x(t) < \dfrac{k}{2^n}, \quad k = 1, 2, 3, \ldots, n2^n, \\ n & \text{for } x(t) \geqslant n. \end{cases}$$

Denoting $B_n = \{t : x(t) \geqslant n\}$, we get

$$\sup_{\tau \in T} \int_E \varphi(\lambda |x_n - x|)\, d\mu_\tau \leqslant \sup_{\tau \in T} \int_{B_n} \varphi(\lambda |x_n - x|)\, d\mu_\tau + \sup_{\tau \in T} \int_{E \setminus B_n} \varphi(\lambda |x_n - x|)\, d\mu_\tau.$$

But the following inequalities hold for every $\tau \in T$:

$$J_\lambda = \sup_{\tau \in T} \int_E \varphi(\lambda |x|)\, d\mu_\tau \geqslant \int_{B_n} \varphi(\lambda |x|)\, d\mu_\tau \geqslant \int_{B_n} \varphi(\lambda n)\, d\mu_\tau = \varphi(\lambda n)\, \mu_\tau B_n;$$

hence $\sup_{\tau \in T} \mu_\tau B \leqslant J_\lambda(\varphi(\lambda n))^{-1}$. If we choose n so large that $J_\lambda(\varphi(\lambda n))^{-1} < \delta$, then for such n we have

$$\sup_{\tau \in T} \int_{B_n} \varphi(\lambda |x - n|)\, d\mu_\tau \leqslant \sup_{\tau \in T} \int_{B_n} \varphi(\lambda |x|)\, d\mu_\tau < \varepsilon.$$

On the other hand we know that the family of measures is uniformly bounded. Hence $\varphi(\lambda 2^{-n})\, \mu_\tau(E \setminus B_n) < \varepsilon$ for sufficiently large n. Finally we obtain

$$\sup_{\tau \in T} \int_E \varphi(\lambda |x_n - x|)\, d\mu_\tau < 2\varepsilon$$

for sufficiently large n.

If $x \in \mathfrak{X}_\varphi$ is arbitrary, the proof is obtained splitting x into positive and negative part.

4.3.1. If $\mathfrak{X}_\varphi^* \subset \mathfrak{X}_\psi^*$, then $\|x_i\|_\varphi \to 0$ implies $\|x_i\|_\psi \to 0$ for arbitrary $x_i \in \mathfrak{X}_\varphi^*$.

The proof is obtained applying the closed graph theorem to the injection map of $\mathfrak{X}_\varphi^*$ in $\mathfrak{X}_\psi^*$, because convergence of a sequence $\{x_i\}$ of elements of $\mathfrak{X}_\varphi^*$ in any of the norms $\|\cdot\|_\varphi$ or $\|\cdot\|_\psi$ implies convergence in measure μ_τ for all $\tau \in T$.

4.3.2. If $\mathfrak{X}_\varphi \subset \mathfrak{X}_\psi$, then $\|x_i\|_\varphi \to 0$ implies $\|x_i\|_\psi \to 0$ for arbitrary $x_i \in \mathfrak{X}_\varphi$.

The proof is analogous to that of 4.3.1, taking into consideration the fact that spaces $\mathfrak{X}_\varphi$ and $\mathfrak{X}_\psi$ are complete with respect to norms $\|\cdot\|_\varphi$ and $\|\cdot\|_\psi$, respectively.

4.4. Let the family of measures $\mathfrak{M}$ possess all properties $1^\circ - 5^\circ$. If $\|x_i\|_\varphi \to 0$ implies $\|x_i\|_\psi \to 0$ for arbitrary $x_i \in \mathfrak{X}_\Omega$, then $\psi \overset{l}{\prec} \varphi$.

The proof will be performed indirectly. Let us suppose, $\psi \overset{l}{\prec} \varphi$ does not hold, i.e. for every system of constants k, b, $v_0 > 0$ there exists $v \geqslant v_0$ for which $\psi(kv) \geqslant b\varphi(v)$. Now, we apply the properties of the family of measures. Let c and δ be the constants from the property 5° (b), and let $\varepsilon > 0$ be given. We take $\eta = \varepsilon(\varphi(u))^{-1}$, where u is chosen so large that

$$\text{(i)} \qquad \varepsilon(\varphi(u))^{-1} = \eta < \delta.$$

By property 5°, there exists a set $K \in \mathfrak{S}_0$ such that

$$\text{(ii)} \qquad c\eta \leqslant A(K) \leqslant \eta.$$

The number u may be chosen in such a manner that $\varepsilon < \delta\varphi(u)$ and

$$\text{(iii)} \qquad \psi(\varepsilon u) > \frac{1}{c\varepsilon}\varphi(u).$$

Applying the inequalities (i) – (iii) we obtain

$$\rho_\psi(\varepsilon u \chi_K) = \psi(\varepsilon u) A(K) \geqslant \psi(\varepsilon u) c\eta > \frac{1}{c\varepsilon}\varphi(u) c\eta = 1.$$

By the definition of the norm (*), we have $\|\varepsilon u \chi_K\|_\psi \geqslant 1$. To calculate $\|\varepsilon u \chi_K\|_\varphi$ we apply (i) and (ii). First, we notice that

$$\rho_\varphi\left(\frac{\varepsilon u \chi_K}{\varepsilon}\right) = \varphi(u) A(K) \leqslant \varphi(u)\eta = \varepsilon.$$

Hence, again by the definition of the norm (*), we obtain $\|\varepsilon u \chi_K\|_\varphi \leqslant \varepsilon$. Since ε is arbitrary, one may define a sequence $x_i \in \mathfrak{X}_\Omega$ such that $\|x_i\|_\varphi \to 0$. But $\|x_i\|_\psi \geqslant 1$, and we get a contradiction.

4.5. From 2.5, 4.3 and 4.4 the following theorems are obtained, immediately:

4.5.1. If the family of measures $\mathfrak{M}$ possesses the properties $1^\circ - 5^\circ$, then the following conditions are mutually equivalent:

(α) $\psi \overset{l}{\prec} \varphi$, (β) $\mathfrak{X}^*_\varphi \subset \mathfrak{X}^*_\psi$, (γ) $\mathfrak{X}_\varphi \subset \mathfrak{X}_\psi$,

(δ) $\|x_i\|_\varphi \to 0$ implies $\|x_i\|_\psi \to 0$ for an arbitrary sequence of elements $x_i \in \mathfrak{X}_\Omega$.

4.5.2. If the family of measures possesses the properties $1^\circ - 5^\circ$, then the following conditions are mutually equivalent:

(α) $\varphi \overset{l}{\sim} \psi$, (β) $\mathfrak{X}^*_\psi = \mathfrak{X}^*_\varphi$, (γ) $\mathfrak{X}_\psi = \mathfrak{X}_\varphi$,

(δ) the norms $\|\cdot\|_\varphi$ and $\|\cdot\|_\psi$ are equivalent in the space $\mathfrak{X}_\Omega$.

4.6. Let us suppose that the family of measures $\mathfrak{M}$ is separable and possesses the properties 1°—5°, with property 5°(b) replaced by the following one: there exist constants δ, c ($\delta > 0$, $0 < c \leqslant 1$) such that for any sequence of numbers η_i satisfying the inequalities $0 < \eta_i \leqslant \delta$ for $i = 1, 2, \ldots$ and convergent to 0, one can choose a sequence of pairwise disjoint sets $K_i \in \mathfrak{S}_0$, $i = 1, 2, \ldots$ such that $c\eta_i \leqslant A(K_i) \leqslant \eta_i$ for $i = 1, 2, \ldots$ Then the following conditions are mutually equivalent:

(α) φ satisfies the condition (Δ_2) for large u,

(β) $\mathfrak{X}_\varphi^* = \mathfrak{X}_\varphi$,

(γ) the space $\mathfrak{X}_\varphi^*$ is separable with respect to the norm $\|\cdot\|_\varphi$.

The proof of the implication (α) ⇒ (β) is easy and runs the same lines as in [20] and [22]. (β) ⇒ (γ) follows from 4.2, immediately (compare [7], [9], and also [20], [22]). (γ) ⇒ (α) is proved in the same manner as the respective implication in theorem 3.8 in [22] (see also [16], [21]). The sequence of functions $x_i = v_i \chi_{K_i}$ occuring in the proof is defined in such a manner that the sets $K_i \in \mathfrak{S}_0$, $i = 1, 2, \ldots$, are chosen as in the proof of theorem 4.4.

5. Various authors defined and investigated Orlicz spaces, and also more general modular spaces. For example, in papers [9], [10], [11], [16] and [19] the authors investigated properties of Orlicz spaces and modular spaces defined by means of families of measures. In my previous papers [20], [22] and [24] I considered spaces of strongly summable functions and sequences. In definition of these spaces there were applied two special cases of the family of measures $\mathfrak{M} = \{\mu_\tau\}$, namely, the case of the family of finite, atomless measures in [22], and the case of the family of finite, purely atomic measures in [20] and [24].

5.1. In [22], the set E was equal to the interval $\langle 0, \infty)$, and the topological space T, to the interval $\langle \tau^*, \infty)$, where τ^* is a positive number fixed for a given family of measures. The space $\mathfrak{X}$ was denoted in [22] by X, and its elements were classes of finite, measurable, real-valued functions equivalent with respect to the relation of equality almost everywhere. The family of measures $\mathfrak{M} = \{\mu_\tau\}$, $\tau \in \langle \tau^*, \infty)$, was defined on the σ-algebra $\mathfrak{S}$ of all Lebesgue measurable subsets of $E = \langle 0, \infty)$, and the measures $\mu_\tau \in \mathfrak{M}$ were supposed to be absolutely continuous with respect to the Lebesgue measure. Thus, by the Radon-Nikodym theorem, there exists a function $a(t, \tau)$ defined on the product $\langle 0, \infty) X \langle \tau^*, \infty)$ measurable with respect to the variable t for every $\tau \in \langle \tau^*, \infty)$, $a(t, \tau) \geqslant 0$ for all $t \in \langle 0, \infty)$, $\tau \in \langle \tau^*, \infty)$, satisfying the condition

$$(+) \qquad \mu_\tau A = \int_A a(t, \tau)\, dt$$

for every Lebesgue measurable set $A \subset E$. Then the integral transformation is of the form

$$(++) \qquad \sigma_\varphi(\tau, x) = \int_0^\infty a(t, \tau)\, \varphi(|x(t)|)\, dt,$$

and the space of functions strongly summable to 0 defined and denoted by X_φ^* in [22], is a special case of the space $\mathfrak{X}_\varphi^*$ of strongly $(\mathfrak{M}, \varphi)$-summable elements defined in the present paper. In order that the integral transformation (++) possesses the same properties as the integral transformation defined in the present paper, it is sufficient that the family of measures possesses the properties 1° – 5°

defined in 1.6. Therefore the kernel $a(t, \tau)$ has to satisfy the conditions given in [22], p. 116 and 129. Examples of kernels satisfying these conditions are given in Part 4, of [22]. Thus the theorems obtained in [22] are special cases of those in the present paper.

5.2. The case of purely atomic measures I investigated in [24] and [20]. As set E and topological space T we take the set of natural numbers, and the symbol τ is replaced by n. Measures μ_n, $n \in T$, are defined on the σ-algebra $\mathfrak{S}$ of all subsets of E, thus, there exist a nonnegative matrix $A=(a_{n\nu})$, $n, \nu = 1, 2, \ldots$, such that

$$(+ + +) \qquad \mu_n B = \sum_i a_{n\nu_i} \text{ for } B=\{\nu_i\} \in \mathfrak{S}, \quad \mu_n \emptyset = 0.$$

If the matrix A possesses the properties 3. (a)—(d) given in [20], p. 243, then the family of measures (+ + +) possesses the properties 1°—5° of 1.6. Examples of matrices possessing the above properties are given in Part 3, of [20]. The family of measures (+ + +) was applied in [20] to define the space T_φ^* of strong (A, φ)-summable sequences. A sequence $x=\{t_\nu\}$ is called strongly (A, φ)-summable to 0 if the transformations $\sigma_\varphi(n, x) \equiv \sigma_n^\varphi(x)$ defined by the formula

$$\sigma_n^\varphi(x) = \sum_{\nu=1}^{\infty} a_{n\nu} \varphi(|t_\nu|)$$

possess for some $\lambda>0$ the following two properties $\sigma_n^\varphi(\lambda x) < \infty$ for $n=1, 2, \ldots$, and $\sigma_n^\varphi(\lambda x) \to 0$ as $n \to \infty$. The space T_φ^* is a special case of the space $\mathfrak{X}_\varphi^*$ defined in the present paper, and theorems given in [20] are special cases of those in this paper.

5.3. Let us yet remark that there exists a connection between property 5° (b) of the family of measures given in 1.6, and the condition of equisplittability of the family of measures given in [12], p. 261 and 262.

It is easily seen that in case of purely atomic measures the property 5° corresponds to the condition $(\mathfrak{D}_1)$ given in [20], p. 241. Also, after formulating the property 5° in the language of [18], this property corresponds to the condition $(\mathfrak{D})$ defined in [18], p. 169.

In case of a family of atomless measures, under assumptions on kernel given in [22], the function A_y defined in [22], p. 129, is continuous with respect to y. Theorefore in proofs of theorems in [22], one is applying the property (+) given in [22], p. 129, in place of the inequality from the property 5° (b).

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