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SOME INEQUALITIES RELATED TO EULER'S THEOREM $R \geq 2r$

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1. Let I denote the incenter of the triangle $\Delta = ABC$ and let AI, BI, CI cut the circumcircle of ABC in A', B', C', respectively. If the angles of ABC are α, β, γ, it is easily verified that the angles of $\Delta' = A'B'C'$ are

$$90^\circ - \frac{1}{2}\alpha, \quad 90^\circ - \frac{1}{2}\beta, \quad 90^\circ - \frac{1}{2}\gamma.$$

We may call Δ' the first derived triangle of ABC. Then if Δ' is the first derived triangle of Δ', it follows that the angles of Δ'' are

$$45^\circ + \frac{1}{4}\alpha, \quad 45^\circ + \frac{1}{4}\beta, \quad 45^\circ + \frac{1}{4}\gamma.$$

We may define the n-th derived triangle $\Delta^{(n)}$ of ABC recursively as the first derived triangle of $\Delta^{(n-1)}$. If $\alpha^{(n)}$, $\beta^{(n)}$, $\gamma^{(n)}$ are the angles of $\Delta^{(n)}$ we have

$$\alpha^{(n)} = \frac{2^n - (-1)^n}{3 \cdot 2^{n-1}}\, 90^\circ + \frac{(-1)^n}{2^n}\alpha \tag{1}$$

with similar formulas for $\beta^{(n)}$, $\gamma^{(n)}$. This is easily proved by induction. Indeed, assuming (1), we get

$$\alpha^{(n+1)} = 90^0 - \frac{1}{2}\alpha^{(n)} = \left(1 - \frac{2^n - (-1)^n}{3 \cdot 2^n)}\right) 90^\circ + \frac{(-1)^{n+1}}{2^{n+1}}\alpha$$

$$= \frac{2^{n+1} - (-1)^{n+1}}{3 \cdot 2^n}\, 90^\circ + \frac{(-1)^{n+1}}{2^{n+1}}\alpha.$$

It follows from (1) that $\Delta^{(n)}$ is equilateral for any fixed n if and only if Δ is equilateral.

All the triangles $\Delta^{(n)}$ evidently have a common circumcircle, namely the circumcircle of Δ.

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It is evident from (1) that as n becomes large the n-th triangle $\Delta^{(n)}$ becomes more nearly equilateral. Thus if R is the radius of the circumscribed circle of Δ and $r^{(n)}$ is the radius of the inscribed circle of $\Delta^{(n)}$, we have

$$\lim_{n=\infty} r^{(n)} = \frac{1}{2} R. \tag{2}$$

Moreover, by Euler's theorem,

$$r^{(n)} \leq \frac{1}{2} R \qquad (r=1, 2, 3, \ldots). \tag{3}$$

We shall now show that

$$r \leq r' \tag{4}$$

with equality if and only if Δ is equilateral. As an immediate corollary of (4) we have

$$r^{(n)} \leq r^{(n+1)} \qquad (n=1, 2, 3, \ldots) \tag{5}$$

with equality if and only if Δ is equilateral.

To prove (4) we recall that [1, p. 192]

$$r = 4R \sin\frac{1}{2}\alpha \sin\frac{1}{2}\beta \sin\frac{1}{2}\gamma. \tag{6}$$

Since the angles of Δ' are

$$\frac{1}{2}(\beta+\gamma),\ \frac{1}{2}(\gamma+\alpha),\ \frac{1}{2}(\alpha+\beta),$$

it follows that

$$r' = 4R \sin\frac{1}{4}(\beta+\gamma) \sin\frac{1}{4}(\gamma+\alpha) \sin\frac{1}{4}(\alpha+\beta). \tag{7}$$

Thus (4) is equivalent to

$$\sin\frac{1}{2}\alpha \sin\frac{1}{2}\beta \sin\frac{1}{2}\gamma \leq \sin\frac{1}{4}(\beta+\gamma) \sin\frac{1}{4}(\gamma+\alpha) \sin\frac{1}{4}(\alpha+\beta). \tag{8}$$

Using the formulas for $\sin 2x$ and $\sin(x+y)$ this reduces to

$$8 \tan\frac{1}{4}\alpha \tan\frac{1}{4}\beta \tan\frac{1}{4}\beta \tag{9}$$

$$\leq \left(\tan\frac{1}{4}\beta + \tan\frac{1}{4}\gamma\right)\left(\tan\frac{1}{4}\gamma + \tan\frac{1}{4}\alpha\right)\left(\tan\frac{1}{4}\alpha + \tan\frac{1}{\beta}\beta\right).$$

For brevity put

$$x = \tan\frac{1}{4}\alpha,\ y = \tan\frac{1}{4}\beta,\ z = \tan\frac{1}{4}\gamma.$$

Then we must show that

$$8xyz \leq (y+z)(z+x)(x+y)$$

or, what is the same thing,

(10) $$6xyz \leq \sum x^2 y.$$

Since

$$\sum x \sum xy = \sum x^2 y + 3xyz,$$

(10) may be replaced by

(11) $$9xyz \leq \sum x \sum xy.$$

This inequality is valid for all non-negative x, y, z with equality only when $x = y = z$.

This evidently completes the proof of (4).

2. We can prove (4) more rapidly by making use of the following result [1, p. 200]. If H is the orthocenter of ABC then

(12) $$\overline{HI}^2 = 4R^2\left(8\sin^2\frac{1}{2}\alpha\sin^2\frac{1}{2}\beta\sin^2\frac{1}{2}\gamma - \cos\alpha\cos\beta\cos\gamma\right).$$

It is easily verified that I is the orthocenter of $A'B'C'$. Hence, applying (12) to the triangle $A'B'C'$, we get

$$\overline{II'}^2 = 4R^2\left\{8\sin^2\frac{1}{4}(\beta+\gamma)\sin^2\frac{1}{4}(\gamma+\alpha)\sin^2\frac{1}{4}(\alpha+\beta) - \sin\frac{1}{2}\alpha\sin\frac{1}{2}\beta\sin\frac{1}{2}\gamma\right\},$$

where I' denotes the incenter of $A'B'C'$. Making use of (6) and (7), this reduces to

$$\overline{II'}^2 = 2r'^2 - Rr.$$

Since

$$\overline{II'}^2 = R^2 - 2Rr,$$

we have

(13) $$2r'^2 = R^2 - Rr = R(-r)$$

and therefore

$$2r'^2 \geq Rr \geq 2r^2$$

with equality only when $R = 2r$.

Another application of (12) may be noted. If 0 is the circumcenter of ABC and 0_1, 0_2, 0_3 the mid points of the sides of ABC, then

$$00_1 = R\cos\alpha,\ 00_2 = R\cos\beta,\ 00_3 = R\cos\gamma.$$

It follows at once from (12) that

(14) $$00_1 \cdot 00_2 \cdot 00_3 \leq \frac{1}{2}Rr^2$$

with equality only when ABC is equilateral.

3. Let K denote the area of Δ and K' the area of Δ'. Also let s denote the semiperimeter of Δ and s' the semiperimeter of Δ'. We shall show that

(15) $$K \leq K'$$

and

(16) $$s \leq s'.$$

Moreover in each case there is equality if and only if ABC is equilateral.

Since

$$K = 2R^2 \sin\alpha \sin\beta \sin\gamma,$$

it is clear that (16) is equivalent to

$$\sin\alpha \sin\beta \sin\gamma \leq \sin\frac{1}{2}(\beta+\gamma) \sin\frac{1}{2}(\gamma+\beta) \sin\frac{1}{2}(\alpha+\beta).$$

This is the same as

$$8\prod \sin\frac{1}{2}\alpha \cos\frac{1}{2}\alpha \leq \prod\left(\sin\frac{1}{2}\beta \cos\frac{1}{2}\gamma + \cos\frac{1}{2}\beta \sin\frac{1}{2}y\right)$$

which is equivalent to

(17) $$8\prod \tan\frac{1}{2}\alpha \leq \prod\left(\tan\frac{1}{2}\beta + \tan\frac{1}{2}\gamma\right).$$

If we put

$$x = \tan\frac{1}{2}\alpha,\quad y = \tan\frac{1}{2}\beta,\quad z = \tan\frac{1}{2}\gamma,$$

(17) becomes

$$8\,xyz \leq (y+z)(z+x)(x+y),$$

which we have already encountered above.

It may be of interest to mention the following result. Since

$$A'B' = 2R\sin\left(90° - \frac{1}{2}\alpha\right) = 2R\cos\frac{1}{2}\alpha,$$

it follows that

$$B'C'\cdot C'A'\cdot A'B' = 8R^3\cos\frac{1}{2}\alpha\cos\frac{1}{2}\beta\cos\frac{1}{2}\gamma = (a+b+c)R^2.$$

Combining this with

$$K' = \frac{B'C'\cdot C'A'\cdot A'B'}{4R},$$

we get

(18) $$K' = \frac{1}{2}Rs.$$

To prove (16) we use

$$s = R(\sin\alpha + \sin\beta + \sin\gamma).$$

Then (16) is equivalent to

(19) $$\sum \sin\alpha \leqslant \sum \sin\frac{1}{2}(\beta+\gamma).$$

Since

$$\sum \sin\alpha = 4\prod \cos\frac{1}{2}\alpha$$

(19) may be replaced by

$$\prod \cos\frac{1}{2}\alpha \leq \prod \cos\frac{1}{4}(\beta+\gamma).$$

This in turn may be replaced by

$$\prod\left(\cos^2\frac{1}{4}\alpha-\sin^2\frac{1}{4}\alpha\right)\leq\prod\left(\cos\frac{1}{4}\beta\cos\frac{1}{4}\gamma-\sin\frac{1}{4}\beta\sin\frac{1}{4}\gamma\right)$$

or

$$\prod\left(1-\tan^2\frac{1}{4}\alpha\right)\leq\prod\left(1-\tan\frac{1}{4}\beta\tan\frac{1}{4}\gamma\right). \tag{20}$$

If we put

$$x=\tan\frac{1}{2}\alpha,\quad y=\tan\frac{1}{4}\beta,\quad z=\tan\frac{1}{4}\gamma,$$

(20) becomes

$$\prod(1-x^2)\leq\prod(1-yz), \tag{21}$$

with $0\leq x\leq 1$, $0\leq y\leq 1$, $0\leq z\leq 1$.

Now

$$(1-y^2)(1-z^2)\leq(1-yz)^2 \tag{22}$$

since this is equivalent to

$$2yz\leq y^2+z^2;$$

moreover equality occurs only when $y=z$. Clearly (21) is an immediate corollary of (22).

If $K^{(n)}$ denotes the area of $\Delta^{(n)}$ and $s^{(n)}$ the semiperimeter of $\Delta^{(n)}$ it follows at once from (15) and (16) that

$$K^{(n)}\leq K^{(n+1)}\qquad(n=1,2,3,\ldots) \tag{23}$$

and

$$s^{(n)}\leq s^{(n+1)}\qquad(n=1,2,3,\ldots) \tag{24}$$

with equality only when ABC is equilateral.

4. Let r_a, r_b, r_c denote the radii of the escribed circles of ABC and r_a', r_b', r_c' the radii of the escribed circles of $A'B'C'$. We shall show that

$$r_a\leq r_a' \tag{25}$$

provided $\alpha\leq 60°$. Since [1, p. 193]

$$r_a=4R\sin\frac{1}{2}\alpha\cos\frac{1}{2}\beta\cos\frac{1}{2}\gamma,$$

it follows that

$$r_a'=4R\sin\frac{1}{4}(\beta+\gamma)\cos\frac{1}{4}(\gamma+\alpha)\cos\frac{1}{4}(\alpha+\beta).$$

Thus (25) is equivalent to

$$2\sin\frac{1}{4}\alpha\cos\frac{1}{4}\left(\cos^2\frac{1}{4}\beta-\sin^2\frac{1}{4}\beta\right)\left(\cos^2\frac{1}{4}\gamma-\sin^2\frac{1}{4}\gamma\right)$$

$$\leq\left(\sin\frac{1}{4}\beta\cos\frac{1}{4}\gamma+\cos\frac{1}{4}\beta\sin\frac{1}{4}\gamma\right)\left(\cos\frac{1}{4}\gamma\cos\frac{1}{4}\alpha-\sin\frac{1}{4}\gamma\sin\frac{1}{4}\alpha\right)$$

$$\cdot\left(\cos\frac{1}{4}\alpha\cos\frac{1}{4}\beta-\sin\frac{1}{4}\alpha\sin\frac{1}{4}\beta\right),$$

or what is the same thing

$$2\tan\frac{1}{4}\alpha\left(1-\tan^2\frac{1}{4}\beta\right)\left(1-\tan^2\frac{1}{4}\gamma\right)$$
$$\leq\left(\tan\frac{1}{4}\beta+\tan\frac{1}{4}\gamma\right)\left(1-\tan\frac{1}{4}\gamma\tan\frac{1}{4}\alpha\right)\left(1-\tan\frac{1}{4}\alpha\tan\frac{1}{4}\beta\right).$$

If we put

$$x=\tan\frac{1}{4}\alpha,\ y=\tan\frac{1}{4}\beta,\ z=\tan\frac{1}{4}\gamma$$

the last inequality becomes

$$2x(1-y^2)(1-z^2)\leq(y+z)(1-zx)(1-xy). \tag{26}$$

Now

$$\tan\frac{1}{4}60^\circ=\sqrt{\frac{1-\cos 30^\circ}{1+\cos 30^\circ}}=2-\sqrt{3}.$$

Hence if $\alpha\leq 60^\circ$ it follows that $x\leq 2-\sqrt{3}$ Also it is easily verified that

$$x^2-4x+1\geq 0$$

when $x\leq 2-\sqrt{3}$. It follows that

$$2x(1+x)\leq(1-x)(1-x^2).$$

Since

$$\frac{1-x}{1+x}=\tan\frac{1}{4}(\beta+\gamma)=\frac{y+z}{1-yz}$$

the last inequality may be replaced by

$$2x(1-yz)\leq(y+z)(1-x^2).$$

This in turn implies

$$2x(1-yz)(1-y^2)(1-z^2)\leq(y+z)(1-y^2)(1-y^2)(1-yz)$$
$$\leq(y+z)(1-yz)(1-zx)(1-xy),$$

by (21). Therefore

$$2x(1-y^2)(1-z)^2\leq(y+z)(1-xy)(1-xz)$$

which is identical with (26).

5. It is not difficult to show that

$$A'I+B'I+C'I\leq A''I'+B''I'+C''I'. \tag{27}$$

Indeed since

$$A'I=2R\sin\frac{1}{2}\alpha$$

and

$$\sum\sin\frac{1}{2}\alpha=1+4\prod\sin\left(45^\circ-\frac{1}{4}\alpha\right),$$

(27) is equivalent to

$$\prod\sin\left(45^\circ-\frac{1}{4}\alpha\right)\leq\prod\sin\left(45^\circ-\frac{1}{8}(\beta+\gamma)\right). \tag{28}$$

The proof of (28) is similar to the proof of (8).

It would be of interest to know whether

$$AI+BI+CI \leq A'I'+B'I'+C'I'. \tag{29}$$

Since

$$AI = 4R \sin\frac{1}{2}\beta \sin\frac{1}{2}\gamma,$$

(29) is equivalent to

$$\sum \sin\frac{1}{2}\beta \sin\frac{1}{2}\gamma \leq \sum \sin\frac{1}{4}(\alpha+\beta) \sin\frac{1}{4}(\alpha+\gamma). \tag{30}$$

We remark that

$$AI \cdot BI \cdot CI = 4Rr^2 \tag{31}$$

and

$$A'I \cdot B'I \cdot C'I = 2R^2 r. \tag{32}$$

6. Summary. The following inequalities are proved.

$$r \leq r', \tag{4}$$

$$K \leq K', \tag{15}$$

$$s \leq s', \tag{16}$$

$$r_a \leq r_a' \qquad (\alpha \leq 60^\circ). \tag{25}$$

For each of the first three inequalities there is equality if and only if the t iangle is equilateral.

Added in proof

Replacing α, β, γ by $180^\circ - 2\alpha$, $180^\circ - 2\beta$, $180^\circ - 2\gamma$, (30) reduces to

$$\sum \cos\beta \cos\gamma \leq \sum \sin\frac{1}{2}\beta \sin\frac{1}{2}\gamma. \tag{33}$$

Since (A. Bager, *A family of goniometric inequalites*, Publications de la Faculté d'électrotechique de l'Université à Belgrade, Série: Math. et phys. no 339 (1971), pp. 5—25)

$$\sum \cos\beta \cos\gamma = \frac{r^2+s^2-4R^2}{4R^2},$$

we may replace (33) by

$$r^2+s^2-4R^2 \leq R\sum AI. \tag{34}$$

Thus (29) is equivalent to (34).

REFERENCE

[1] E. W. Hobson, *A Treatise on Plane and Advanced Trigonometry*, Dover, New York, 1957.

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