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MECHANICS OF GROWING MATERIALS WITH MEMORY

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I. Introduction

The objective of this study is to develop a constitutive theory capable of describing the effect of mechanical loads on the form and stress distribution in a growing body. Explicitly, if the final form and growth history of a growing material with memory are known, then what will the final form of the body be if it experiences mechanical loads during the growth process. Conversely, if the present state of stress and past history are known for the growing body, then what is the distribution of the stresses if the body experienced deformations during its growth. In other words, if some base history determining the form or state of stress is known for a growing body, then what are the effects of perturbing this base history.

A major application of a theory of growing materials with memory is to biological growth. For example, a major motivation for the development of the constitutive relations in this paper is to determine the mechanical and growth response of human bones to external loading situations. The solution to this medically significant problem may someday find use in treating certain skeletal system problems. Presently, however, there is a dearth of experimental data on mechanically loaded growing materials. Most medical investigations have been of a qualitative nature and are very helpful in formulating general constitutive equations capable of describing the observed effects, but lack the data necessary for the formulation and solution of boundary value problems. A large literature exists in which the mechanical properties of preserved biological members are described, but this literature has only minor validity in regard to in-vivo growing bodies.

The collagen structure of the skeletal system is analogous to the cellulose structure of wood. Both wood and bone have similar growth patterns, orthotropic symmetry, and piezoelectric properties. The application of a theory of growing materials to wood would be to attempt to control density, hardness and strength by means of the application of external mechanical loads during growth.

There are various categories of growing bodies, both organic and inorganic, and processes where mass is gained and processes where mass is lost. It is worthwhile to note that many existing special theories fall within the general framework of the theory of growing materials. For example, theories of crystal growth, corrosion, diffusion, penetration of one solid by another, melting, formation and flow of glaciers. etc.

II. Balance Equations

The growing body will be denoted by B, where B is the object called the growing body. Let B occupy a portion of a three dimensional Euclidean space. The portion of this space occupied by B and described in a rectangular Cartesian coordinate system, is called the form of B. The form of B is denoted by $\underset{\sim}{B}$.

The mass of B and the balance equations B must satisfy, can be developed in very general terms. For example, in terms of Lebesque integrals. General treatments would be applicable to a wide range of bodies with certain discontinuous properties. This study is concerned with well behaved growing bodies with continuous mass distributions. Thus Riemann integrals are of sufficient generality for these bodies. The mass will be defined as,

$$m = \int_{\underset{\sim}{B}} \rho \, dV \tag{2.1}$$

where the concepts of density ρ, and volume V, of B in the from $\underset{\sim}{B}$ are taken as intuitive.

The mass balance equation can now be written in terms of (2.1)

$$\frac{D}{Dt} \int_{\underset{\sim}{B}} \rho \, dV = - \int_{\partial \underset{\sim}{B}} C \, dA + \int_{\underset{\sim}{B}} Q \, dV \tag{2.2}$$

The first term on the right hand side of equation (2.2) is a measure of the flux of mass across the boundary of $\underset{\sim}{B}$. The second term measures the mass increase due to sources in the interior of $\underset{\sim}{B}$. In keeping with the objectives stated in the introduction, the surface mass flux term will be neglected. Neglect of this term excludes this theory from being applicable to any but internal mass source phenomena. Applying this restriction and the identity

$$\frac{D}{Dt} dV = \nabla \dot{x} \, dV \tag{2.3}$$

where x is the position vector of a particle in $\underset{\sim}{B}$, yields,

$$\int_{\underset{\sim}{B}} \left[\frac{D\rho}{Dt} dV + \rho \frac{D}{Dt} dV \right] - \int_{\underset{\sim}{B}} Q \, dV = 0 \tag{2.4}$$

whence

$$\frac{D\rho}{Dt} + \rho \nabla \dot{x} = Q. \tag{2.5}$$

In the transition from (2.2) to (2.5) the standard mathematical assumptions used in deriving the ordinary balance equations must hold. It is possible to define Q as the increase in mass per unit volume. This increase in mass Q will be called the growth intensity.

The standard equilibrium equation

$$S_{ij,i} + \rho f_i = 0 \tag{2.6}$$

applies to growing materials, were $\underset{\sim}{S}$ is the stress and $\underset{\sim}{f}$ the body force.

III. Constitutive Equations

Let $\underset{\sim}{x}$ be the position vector of a generic particle $\underset{\sim}{X}$ of B. If B is growing or deforming, or both, then the growth or deformation can be described by specifying the position of a generic particle $\underset{\sim}{x}$ in time as a function of $\underset{\sim}{X}$. Thus

(3.1) $$\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{X}, t).$$

The deformation gradient is defined as

(3.2) $$\underset{\sim}{F} = \frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{X}}$$

and can be decomposed by means of a polar decomposition

(3.3) $$\underset{\sim}{F} = \underset{\sim}{R}\underset{\sim}{U}$$

where $\underset{\sim}{R}$ is a pure rotation and $\underset{\sim}{U}$ is a pure stretch. The stretching $\dot{U} = d$ was used by Hsu [1] in developing constitutive relations for growing elastic bodies. The general constitutive relation stated in [1] is

(3.4) $$\underset{\sim}{d} = f(t, \rho, \dot{\rho}, \underset{\sim}{S}, \dot{\underset{\sim}{S}})$$

and a linear form of (3.4) for materials with constant density was used to solve some boundary value problems.

To determine the present state of stress in a growing material with memory, let this stress state be dependent upon the entire past histories of growth and deformation. We define a material of this type to be a simple growing material. Mathematically this statement can be expressed as a functional relationship,

(3.5) $$\underset{\sim}{\sigma}(t) = \underset{\sim}{\mathfrak{F}}\,[\underset{\sim}{E}(\tau)]_{\tau=-\infty}^{\tau=t}$$

where $\underset{\sim}{\sigma}$ is the rotated or Kirchoff stress tensor and $\underset{\sim}{E}$ is the Green strain tensor

(3.6) $$\underset{\sim}{E} = \frac{1}{2}\,[\underset{\sim}{F}\underset{\sim}{F}^{\dagger} - \underset{\sim}{I}].$$

Thus $\underset{\sim}{E}$ is descriptive of the growth of the material as well as its deformation, and any specific constitutive law will take into account the fact that growth may cause internal stresses in the material. If the body under consideration was created at some time, say, $\tau = 0$, then we may write

(3.7) $$\underset{\sim}{\sigma}(t) = \underset{\sim}{\mathfrak{F}}\,[\underset{\sim}{E}(\tau),\ t]_{\tau=0}^{\tau=t}$$

where the stress is explicitly dependent upon the present time t. This allows the consideration of the ageing of growing materials, or the change of material properties with time.

The composite growth and deformation history $\underset{\sim}{E}(\tau)$ can be decomposed into a base history $\underset{\sim}{E}_0(\tau)$ due to growth and normal service loads and a perturbation history $\underset{\sim}{e}(\tau)$. For example, the base history of a human femur could be its growth history and the deformations experienced by the femur in walking, swimming, etc. For this base history there is experimental evidence concerning the state of stress in various materials. In fact the base history may be interpreted as the normal mechanical life of a biological material.

For a state of isotropic growth, no deformation, and constant growth rate

(3.8) $$\underset{\sim}{E}_0(\tau) = \underset{\sim}{K}(\tau)$$

where $\underset{\sim}{K}(\tau)$ is a linear function of time and yields equal extensions in the principal directions. However, for materials such as bone and wood growth is anisotropic, the growth rate is a complex function of time and deformations are produced by the growth process without external influences. Therefore, in general $\underset{\sim}{K}(\tau)$ is an anisotropic tensor and nonlinear in time. In any case, the base history for growth without external influences can be experimentally determined for many materials.

Now let the material be deformed in a manner so as to perform a perturbation on the base history and let the history of these perturbations be denoted by $\underset{\sim}{e}(\tau)$. Note that no restriction is placed upon the size of the perturbation.

The constitutive relation (3.7) may now be written as

$$\underset{\sim}{\sigma}(t) = \underset{\sim}{\mathfrak{F}}\,[\underset{\sim}{E}_0(\tau),\ \underset{\sim}{e}(\tau),\ t]_{\tau=0}^{\tau=t} \tag{3.9}$$

where $\underset{\sim}{E}_0$ is measurable and $\underset{\sim}{e}$ is chosen in a manner consistent with the objectives of the experiment, or observed if a natural process is under consideration

Let us assume that the growing material with memory lends itself to description by means of a bilinear functional, i.e. the functional is linear in $\underset{\sim}{E}_0$ and $\underset{\sim}{e}$. In this case an extension of a representation theorem on bilinear functionals in [2] can be applied and an exact representation of the constitutive relation (3.9) is obtained

$$\underset{\sim}{\sigma}(t) = \int_0^t \int_0^t \underset{\sim}{K}(t, \tau_1, \tau_2)\, \underset{\sim}{E}_0(\tau_1)\, \underset{\sim}{e}(\tau_2)\, d\tau_1\, d\tau_2 \tag{3.10}$$

In this constitutive equation the effects of the perturbation are coupled with the effects of the base history. This is desirable since in practice the material can not separate out effects due to the external deformations and the deformations due to growth or normal service.

If we are dealing with a process where the effects due to the base history can be separated from the perturbation history, then the constitutive representation can be simplifed. As a special case consider a deformation process (history) where

$$\underset{\sim}{e}(\tau) = \underset{\sim}{E}(\tau) - \underset{\sim}{E}_0(\tau) \tag{3.11}$$

defines the perturbation history. For this type of process the basic constitutive relation (3.7) becomes

$$\underset{\sim}{\sigma}(t) = \underset{\sim}{\mathfrak{F}}[\underset{\sim}{E}_0(\tau) + \underset{\sim}{e}(\tau),\ t]_{\tau=0}^{\tau=t}. \tag{3.12}$$

If we assume that the growing material has linear history dependent properties then a theorem of Volterra may be applied and (3.12) becomes [3]

$$\underset{\sim}{\sigma}(t) = \int_0^t \underset{\sim}{K}(t, \tau)\, \underset{\sim}{E}_0(\tau)\, d\tau + \int_0^t \underset{\sim}{L}\,(\underset{\sim}{E}_0(\tau), \tau)\, \underset{\sim}{e}(\tau)\, d\tau. \tag{3.13}$$

Note that if no perturbations are applied to the material then (3.13) reduces to

$$\underset{\sim}{\sigma}(t) = \int_0^t \underset{\sim}{K}(t, \tau)\, \underset{\sim}{E}_0(\tau)\, d\tau \tag{3.14}$$

which is the expression for the state of stress in a growing, ageing material in unperturbed growth. If we have no knowledge of the base history of the material, then a linear functional in the history $\underset{\sim}{E}(\tau)$ should be used.

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