PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 12 (26), 1971. pp. 115—129

NEARLY UNIFORM CONVERGENCE AND INTERCHANGE OF LIMITS

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(Received June 29, 1971)

1. Introduction

Sometime in one's undergraduate career, one sees the statement that the uniform limit of a sequence of continuous funtions is continuous. Clearly, this is a statement regarding interchange of limits; i.e.

$$\lim_{x\to x_0} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \lim_{x\to x_0} f_n(x).$$

E. H. Moore [4] developed a generalized idea of convergence and showed that for a function $F(x, y)$ of two variables, if we assume that $F(\ , y)$, $F(x,\ )$ converge uniformly in y, x respectively as x converges to x_0 and y converges to y_0, then the limits may be interchanged; i.e.,

$$\lim_{x\to x_0} \lim_{y\to y_0} F(x, y) = \lim_{y\to y_0} \lim_{x\to x_0} F(x, y).$$

Helsel [1] shows that for real valued functions of a real variable, the condition of uniform convergence can be weakened to obtain a condition which is in a certain sense necessary and sufficient for the interchange of limits.

These results can easily be generalized to include the interchange of generalised limits in a uniform space — sometimes defined as the most general space in which the idea of uniform convergence makes sense. Because the proofs of these results require only the definitions of generalized limit, uniform space, (nearly) uniform convergence, and the idea of the proof that the uniform limit of a sequence of continuous functions is continuous, it is believed that the students should see the proof in the general situation so as to obtain a deeper insight into the nature of the interchange of limits.

2. Definitions and Notation

If X is a non-empty set, a family $\mathcal{U}$ of subsets of $X\times X$ is called a uniformity for X if:

1) each $U\in\mathcal{U}$ contains the diagonal $\{(x, x): x\in X\}$

2) $V\supset U\in\mathcal{U}\to V\in\mathcal{U}$

3) $U_1, U_2\in\mathcal{U}\to\exists U_3\in\mathcal{U}$ such that $U_3\subset U_1\cap U_2$

4) $U \in \mathcal{U} \rightarrow U^{-1} = \{(y, x): (x, y) \in U\} \in \mathcal{U}$

5) $U \in \mathcal{U} \rightarrow \exists V \in \mathcal{U}$ such that $V^2 \subset U$ where $V^2 = V \circ V = \{(x, y): \exists z \in X$ such that $(x, z) \in V$ and $(z, y) \in V\}$. Thus $U \in \mathcal{U} \rightarrow \exists W \in \mathcal{U}$ such that $W^3 \subset U$.

The U's in $\mathcal{U}$ play the role analogous to that of the positive ε's in a metric space. $(x, y) \in U$ is the statement analogous to $d(x, y) < \varepsilon$. The V in (5) then corresponds to $\varepsilon/2$; i.e. $(x, z) \in V$, $(z, y) \in V \rightarrow (x, y) \in U$ corresponds to $d(x, z) < \varepsilon/2$, $d(z, y) < \varepsilon/2 \rightarrow d(x, y) < \varepsilon$. The sets $U[x] = \{y: (x, y) \in U\}$ play the role of spheres centered at x; thus we get a natural concept of limit.

Recall that a non-empty set D is said to be directed if there is a transitive binary relation $\prec$ defined on D such that for every two points $x, y \in D$, there is a $z \in D$ for which $x \prec z$ and $y \prec z$.

Observe that the set of natural numbers and the set of real numbers are each directed by both $\leq$ and $<$. If $A \subset R$ and $a \in R$, the $A - \{a\}$ is directed by

$$x \prec y \equiv |y - a| \leq |x - a|, \; x, y \in A, \; x \neq a \neq y.$$

In fact this directing relation can be generalized to any metric space. It is called the directing relation which describes x converging to a.

A net is a function whose domain is a directed set. Thus a sequence is a net. A function with real domain (or any subset of a metric space as its domain) can be thought of as a net in many different ways; each $a \in X$ gives us a directing relation associated with $x \rightarrow a$.

Is f is a net with domain D we sometimes write $\{a_\nu\}_{\nu \in D}$ instead of $f: D \rightarrow X$. Thus net notation resembles sequential notation and in fact we paraphrase the definition of sequential convergence (Cauchyness) to obtain the definition for net convergence (Cauchyness). It can be shown that every Cauchy net in a complete metric space converges to a point of the space. (See McShane [2].)

Since the $U \in \mathcal{U}$ for a uniform space takes the place of $\varepsilon > 0$ for a metric space we know how to define Cauchy net in a uniform space. Thus the concept of completeness in a uniform space makes sense; in fact we define a complete uniform space as one in which each Cauchy net converges.

If D is a directed set and $\nu_0 \in D$, then $D_0 = \{\nu: \nu \succ \nu_0\}$ will be called a tail of D. Clearly each tail of D is directed so that any net on D is a net on each tail of D. Also, the limit of the net on D (if it exists) will equal the limit of the corresponding net on any tail of D. Thus if f is a net on some tail D_0 of D we shall write $\lim_{x \in D} f(x)$ for $\lim_{x \in D_0} f(x)$ even though $f(x)$ may not be defined for $x \in D \setminus D_0$. We can do this since, if the limits of two such "tail nets" exist, these limits must be equal (in a T_2 space).

Let D_1, D_2 be directed sets with directing relations $\succ_1$ and $\succ_2$ respectively. Then $D_1 \times D_2$ is a directed set when we define $(d_1, d_2) \succ (d_1^*, d_2^*) \equiv [d_1 \succ_1 d_1^*$ and $d_2 \succ_2 d_2^*]$. Any function f defined on $D_1 \times D_2$ is a net. In addition, if $x_0 \in D_1$, $y_0 \in D_2$ then $f(x_0, y)$ is a net (consider its domain as D_2) and $f(x, y_0)$ is a net (consider its domain as D_1).

Hence, if the range is a uniform space, we may consider the existence of

1) $\lim_{y} f(x_0, y)$
2) $\lim_{x} f(x, y_0)$
3) $\lim_{x,y} f(x, y)$

Also, if $\lim_{x} f(x, y_0)$ exists for each $y_0 \in D_2$ (or some tail of D_2), then $\{\lim_{x} f(x, y_0\}_{y_0 \in D_2}$ is a net on D_2 (or that tail of D_2); if $\lim_{y} f(x_0, y)$ exists for all $x_0 \in D_1$ (or some tail of D_1), then $\{\lim_{y} f(x_0, y)\}_{x_0 \in D_1}$ is a net on D_1, (or that tail of D_1). Thus we can also consider the iterated limits.

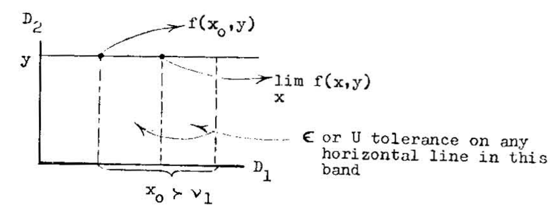
4) $\lim_{x} [\lim_{y} f(x, y)]$
5) $\lim_{y} [\lim_{x} f(x, y)]$

3. Nearly Uniform Convergence

Throughout the rest of this note f will denote a net $f: D_1 \times D_2 \to X$ where $(X, \mathcal{U})$ is a uniform space. The standard concepts of uniform convergence translate into the following.

Definition: $\lim_{x} f(x, y)$ is *uniform* in y (or $f(x, y)$ converges uniformly on D_2) if

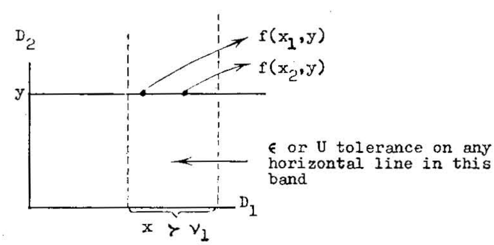
1) $\lim_{x} f(x, y)$ exists for each $y \in D_2$
2) $U \in \mathcal{U} \to \exists \nu_1 \in D_1$ such that $(f(x_0, y), \lim_{x} f(x, y)) \in U$ when $x_0 \succ \nu_1$ and $y \in D_2$.



Definition: $f(x, y)$ is *uniformly Cauchy* on D_2 if

$$U \in \mathcal{U} \to \exists \nu_1 \in D_1 \text{ such that } [f(x_1, y), f(x_2, y)] \in U$$

when $x_1, x_2 \succ \nu_1$ and $y \in D_2$.



We shall find it useful to weaken these concepts as follows:

Definition: $\lim_x f(x, y)$ is *nearly uniform in y* if there is some $\nu_0 \in D_2$ such that $\lim_x f(x, y)$ exists for $y \succ \nu_0$ (i.e., $\lim_x f(x, y)$ exists on some tail of D_2) and

$$* \quad U \in \mathcal{U} \rightarrow \exists (\nu_1, \nu_2) \in D_1 \times D_2 \text{ such that}$$
$$[\lim_x f(x, y_0), f(x^*, y_0)] \in U \text{ when } (x^*, y_0) \succ (\nu_1, \nu_2).$$

Note: W.l.o.g. $\nu_2 \succ \nu_0$.

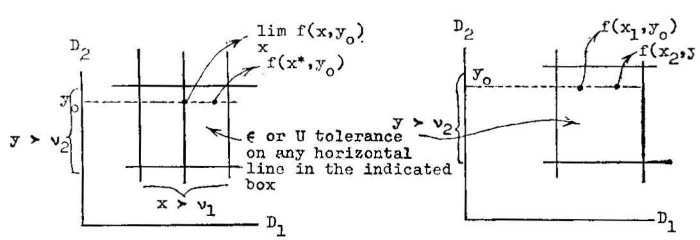
Definition: $f(x, y)$ is *nearly uniformly Cauchy in y* if

$$** \quad U \in \mathcal{U} \rightarrow \exists (\nu_1, \nu_2) \in D_1 \times D_2 \text{ such that } [f(x_1, y_0),$$
$$f(x_2, y_0)] \in U \text{ when } (x_i, y_0) \succ (\nu_1, \nu_2),\ i = 1, 2.$$

If X is a metric space, (*) would read as follows:

$$* \quad \varepsilon > 0 \rightarrow \exists (\nu_1, \nu_2) \in D_1 \times D_2 \text{ such that } d[f(x^*, y_0),$$
$$\lim_x f(x, y_0)] < \varepsilon \text{ when } (x^*, y_0) \succ (\nu_1, \nu_2).$$

A similar restatement holds for **. The reader may find the following illustrations suggestive of what is meant by the above concepts.



nearly uniform in y

nearly uniformly Cauchy in y

N.B. ** does not even say that $\{f(x, y_0)\}_{x \in D_1}$ is Cauchy in x for a fixed y_0 since the ν_2 depends on U. Consider the following:

Example 1: Helsel [1]. Let $f(x, y) = y \sin 1/x$ for (x, y) in the open 1st quadrant. Use directing relations which describe $x \rightarrow 0$, $y \rightarrow 0$. Clearly $f(x, y)$ is nearly uniformly Cauchy in y yet $\{f(x, y)\}_{x>0}$ is not Cauchy nor does $\lim_{x \rightarrow 0} f(x, y_0)$ exist for any $y > 0$.

Of course uniform convergence (Cauchyness) implies nearly uniform convergence (Cauchyness). The following example shows that these "nearly" concepts are indeed weaker.

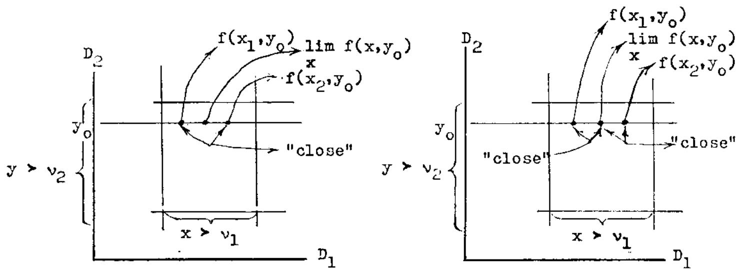
Example 2: (Helsel [1]). Let $f(x, y) = 1/n$ if $x + y = 1/n$ with $y \geq x$, and 0 otherwise on the plane with the origin deleted. Use the natural directing relations which specify $x \rightarrow 0$, $y \rightarrow 0$. $\lim_x f(x, y)$ is nearly uniform in y but not uniform in y.

Since nearly uniform Cauchyness does not imply Cauchyness, the existence of single limits must be assumed before the Cauchy criterion can be established.

Theorem 1: If $\lim_x f(x, y)$ exists on some tail of D_2 (say for $y \succ d_0 \in D_2$), then this limit is *nearly uniform* in y if and only if $f(x, y)$ is *nearly uniformly Cauchy* in y.

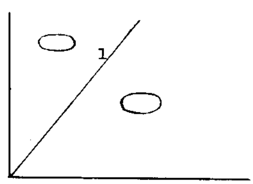
Proof: The standard $\varepsilon/2$ argument is what is needed here. Let $U \in \mathcal{U}$ be given; $\exists$ symmetric $V \in \mathcal{U}$ such that $V^2 \subset U$. If the limit is nearly uniform in y then $\exists (\nu_1, \nu_2) \in D_1 \times D_2$ such that $[f(x^*, y_0), \lim_x f(x, y_0)] \in V$ when $(x^*, y_0) \succ (\nu_1, \nu_2)$. Thus $[f(x_1, y_0), f(x_2, y_0)] \in V \circ V \subset U$ when $(x_i, y_0) \succ (\nu_1, \nu_2)$, $i = 1, 2$. Hence $f(x, y)$ is nearly uniformly Cauchy in y.

Conversely, if $f(x, y)$ is nearly uniformly Cauchy in y, then $\exists (\nu_1, \nu_2) \in \in D_1 \times D_2$ such that $[f(x_i, y_0), f(x_2, y_0) \in V$ when $(x_i, y_0) \succ (\nu_1, \nu_2)$, $i = 1, 2$. Fix $y_0 \in D_2$ such that $y_0 \succ \nu_2$. Then $\exists \nu_1^* \in D_1$ such that $[f(x_2, y_0), \lim_x f(x, y_0)] \in V$ when $x_2 \succ \nu_1^*$. Fixing $\nu_3 \succ \nu_1^*, \nu_1$ we see that $[f(x_1, y_0), \lim_x f(x, y_0)] \in V \circ V \subset U$ when $x_1 \succ \nu_1$. But $y_0 \succ \nu_2$ was arbitrary so this last result holds whenever $(x^*, y_0) \succ (\nu_1, \nu_2)$; i.e. the limit is nearly uniform in y.



Because of the last theorem we should consider another example, which will show us that the $\lim_x f(x, y)$ can exist on some tail of D_2 without being nearly uniform in y.

Example 3: (Helsel [1])



Let $f(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases} \quad x > 0,\ y > 0$

with directing relations so that $x \to 0$, $y \to 0$.

We define "*nearly uniform* (—ly Cauchy)" in x similarly and obtain the analogous results and similar examples.

4. Interchange of Limits

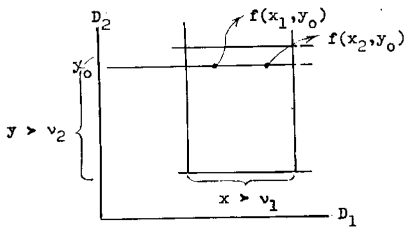
We are now ready to discuss iterated and double limits. The following theorem shows that the "nearly uniform" concepts are intimately related to the existence of the double limit.

Theorem 2: $f(x, y)$ is Cauchy in both variables simultaneously if and only if $f(x, y)$ is nearly uniformly Cauchy in each variable separately. Thus, if the double limit, $\lim_{x,y} f(x, y)$ exists then $f(x, y)$ is nearly uniformly Cauchy in x and in y. Then if either of the single limits exists on some tail, that single limit is nearly uniform.

Proof. Assume $f(x, y)$ is Cauchy in x and y simultaneously. In virtue of the symmetry of the problem, we need only establish the nearly uniform Cauchyness in one of the variables.

$$U \in \mathcal{U} \rightarrow \exists (\nu_1 \nu_2) \in D_1 \times D_2 \text{ such that } [f(x_1, y_1), f(x_2, y_2)] \in U$$
$$\text{when } (x_i, y_i) \succ (\nu_1, \nu_2), \; i = 1,2.$$

Thus $[f(x_1, y_0), f(x_2, y_0)] \in U$ when $(x_i, y_0) \succ (\nu_1, \nu_2)$, $i = 1,2$. This establishes the nearly uniform Cauchyness in y.



Conversely, assume $f(x, y)$ is nearly uniformly Cauchy in x and in y.

$$U \in \mathcal{U} \rightarrow \exists V \in \mathcal{U}$$

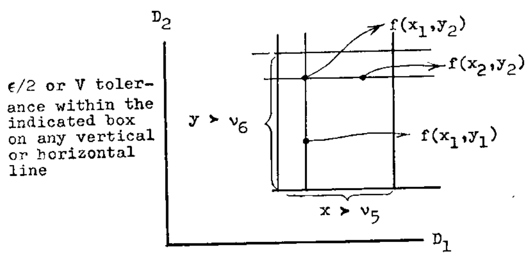
such that $V^2 \subset U$. $\exists (\nu_1, \nu_2) \in D_1 \times D_2$ such that $[f(x_1, y_0), f(x_2, y_0)] \in V$ when

$$(x_i, y_0) \succ (\nu_1, \nu_2), \; i = 1,2.$$

$$\exists (\nu_3, \nu_4) \in D_1 \times D_2 \text{ such that } [f(x_0, y_1), f(x_0, y_2)] \in V$$
$$\text{when } (x_0, y_i) \succ (\nu_3, \nu_4), \; i = 1,2.$$
$$\exists (\nu_5, \nu_6) \succ \text{ both } (\nu_1, \nu_2), (\nu_3, \nu_4).$$

Then $[f(x_1, y_1), f(x_2, y_2)] = [f(x_1, y_1), f(x_2, y_1)] \circ [f(x_2, y_1), f(x_2, y_2)] \in$
$$V \circ V \subset U \text{ when } (x_i, y_i) \succ (\nu_5, \nu_6) \; i = 1,2.$$

Thus $\{f(x, y) : (x, y) \in D_1 \times D_2\}$ is Cauchy.



Note D in the above theorem that the single limits may not exist even though the double limit does in a complete space. Consider the following, for example.

Example 4: (Helsel [1].)

$$f(x, y) = \begin{cases} y \sin 1/x & \text{if } y \geq x > 0 \\ x \sin 1/y & \text{if } x \geq y > 0 \end{cases}$$

where the directing relation specifies that $x \rightarrow 0$, $y \rightarrow 0$.

We now prove the general theorem on interchange of limits. Of course, the iterated limit is a meaningless concept unless the single limit exists on some tail (of D_1 or D_2). Thus the assumptions in the following lemmas and theorem are minimal.

Lemma 3.1; If the single limits exist on respective tails, then nearly uniform convergence in x is equivalent to nearly uniform convergence in y. Thus 1) and 2) in the theorem below are equivalent even in the absence of completeness.

Proof: In virtue of the symmetry of the problem we need only establish the implication in one direction. Assume that $\lim\limits_{x} f(x, y)$ is nearly uniform in y and hence is nearly uniformly Cauchy in y. $U \in \mathcal{U} \to \exists$ a symmetric

$$V \in \mathcal{U} \text{ such that } V^3 \subset U.$$

$$\exists (\nu_1, \nu_2) \in D_1 \times D_2 \text{ such that } [f(x_1, y), f(x_2, y)] \in V \text{ when } (x_i, y) > (\nu_1, \nu_2), \quad i=1,2.$$

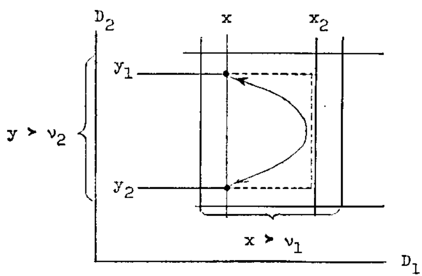
Fix $x_2 > \nu_1$ such that $\lim\limits_{y} f(x_2, y)$ exists.

$$\exists \nu_3 \text{ such that } [f(x_2, y_1), f(x_2, y_2)] \in V \text{ when } y_i > \nu_3, \quad i=1,2$$

since a converging net is a Cauchy net. Let $\nu_4 > \nu_2, \nu_3$. Then if $(x, y_i) > (\nu_1, \nu_4)$ $i=1,2$ we have

$$\begin{aligned}[f(x, y_1), f(x, y_2)] &= [f(x, y_1), f(x_2, y_1)] \\ &\circ [f(x_2, y_1), f(x_2, y_2)] \\ &\circ [f(x_2, y_2), f(x, y_2)] \\ &\in V \circ V \circ V \subset U\end{aligned}$$

Thus the limit is nearly uniformly Cauchy in x and hence nearly uniform in x by Theorem 1.



Example 5:

Letting $f(x, y) = \begin{cases} 1, & x \text{ irrational} \\ 0, & x \text{ rational} \end{cases}$

we see that the above lemma cannot be established for nearly uniform Cauchyness. Note also that, in this example, only one of the single limits exists (use directing relations such that $x \to 0$, $y \to 0$, or $x \to \infty$, $y \to \infty$, etc).

Considering Example 2) again, we see that, even though the concepts of "nearly uniformly convergent" in x and in y are equivalent, the concepts of "uniformly convergent" in x and in y are not equivalent.

Lemma 3.2: If the double limit $\lim_{x,y} f(x, y)$ exists and if one of the single limits exists on a given tail (of D_1, D_2 respectively) the corresponding iterated limit exists and equals the double limit, (even in the absence of completeness).

Proof: Again a standard $\varepsilon/2$ calculation is what we need. By symmetry of the problem we may assume $\lim_x f(x, y)$ exists on some tail of D_2. Note that the existence of the double limit implies the nearly uniform convergence of the single limit. Thus $U \in \mathcal{U} \rightarrow \exists$ a symmetric $V \in \mathcal{U}$ such that $V^2 \subset U$.

$$\exists (\nu_1, \nu_2) \in D_1 \times D_2$$

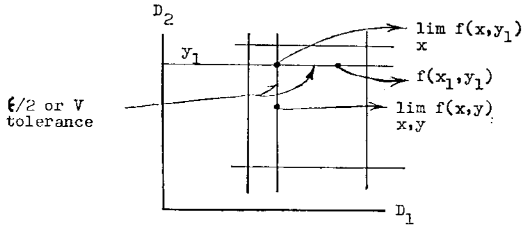
such that

$$[\lim_x f(x, y_1), f(x_1, y_1)] \in V \text{ when } (x_1, y_1) \succ (\nu_1, \nu_2).$$

But $\exists (\nu_3, \nu_4)$ such that $[\lim_{x,y} f(x, y), f(x_1, y_1) \in V$ when $(x_1, y_1) \succ (\nu_3, \nu_4)$. Let $(\nu_5, \nu_6) \succ$ both (ν_1, ν_2), (ν_3, ν_4). Fix $x_1 \succ \nu_5$. Then

$$y \succ \nu_6 \rightarrow [\lim_{x,y} f(x, y), \lim_x f(x, y)] = [\lim_{x,y} f(x, y), f(x_1, y_1)] \circ [f(x_1, y_1),$$

$$\lim_x f(x, y_1)] \in V \circ V \subset U; \text{ i.e. } \lim_{x,y} f(x, y) = \lim_y \lim_x f(x, y).$$



Theorem 3: Let $f: D_1 \times D_2 \rightarrow X$ where $(X, \mathcal{U})$ is a complete uniform space. Assume that the single limits $\lim_x f(x, y)$, $\lim_y f(x, y)$ exist on tails of D_2, D_1 respectively. Then the following are equivalent:

1) $\lim_x f(x, y)$ is nearly uniform in y
2) $\lim_y f(x, y)$ is nearly uniform in x
3) The double limit exists
4) The double limit and both iterated limits exist and all are equal.

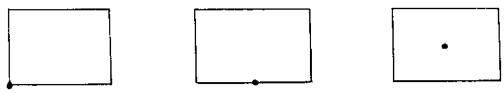
Proof: We have already established that 1) $\rightleftarrows$ 2), and 3) $\rightleftarrows$ 4), and 4) $\rightarrow$ 1) even in the absence of completeness. Thus the only implication yet to be proved is that 1) $\rightarrow$ 3). Assuming 1) we see that both single limits are nearly uniform (since 1) and 2) are equivalent) and hence $f(x, y)$ is nearly uniformly Cauchy in x and in y. It then follows from Theorem 2 that $\{f(x, y)\}_{x, y \in D_1 \times D_2}$ is Cauchy; since the space is assumed to be complete, the limit $\lim_{x,y} f(x, y)$ then exists.

In the above proof we needed the existence of the single limits to obtain the Cauchyness of $\{f(x, y): (x, y) \in D_1 \times D_2\}$ since 1) and 2) in terms of nearly uniform Cauchyness are not equivalent as Example 5) shows.

Thus the double limit can exist only if the single limits are nearly uniform and in this case the iterated limits also exist and all are equal. Again considering Example 3), we see that the iterated limits can exist and be equal without either single limit being nearly uniform. Of course, the double limit cannot exist in this case.

We emphasize the generality of the above work by citing several situations which are included in the theory.

Example 5: Let $f(x, y)$ be a function of two real variables which is defined for points arbitrarily close to (a, b). Depending on what kind of limits we want to consider we consider rectangular "neighborhoods" of (a, b) where (a, b) may be a corner point, side point, or interior point of the rectangle. (See Helsel [1].)



$\lim_{\substack{x \to a^+ \\ y \to b^+}} f(x,y)$ $\qquad$ $\lim_{\substack{x \to a \\ y \to b^+}} f(x,y)$ $\qquad$ $\lim_{\substack{x \to a \\ y \to b}} f(x,y)$

Of course, each side of such a rectangle is a metric space and can be directed, as discussed earlier, so that $x \to a$ and $y \to b$. Note that we don't even need to have $f(x, y)$ defined for all points of the rectangular "neighborhood". All we need is a subset D_1 of R for one side, a subset D_2 of R for the other side such that $f(x, y)$ is defined when $(x, y) \in D_1 \times D_2$. Of course there must exist points of $D_1 \times D_2$ which are arbitrarily close to (a, b). "$\lim_x f(x, y)$ exists on a tail of D_2" translates into "$\lim_{x \to a} f(x, y)$ exists for all y sufficiently close to b" (such that $f(x, y)$ is defined on $D_1 \times \{y\}$).

Example 6: Let $f_n : I \to R$ for $n = 1, 2 \ldots$ where I is an interval (or any metric or uniform space for that matter; we may likewise consider functions with values in a uniform space). Let $f_n \Rightarrow f$ on I and let each f_n be continuous. Then f is likewise continuous.

Proof: Fix $x_0 \in I$. Let D_1 be the set of natural numbers and let D_2 be directed with the relation which specifies $x \to x_0$. Consider $F : D_1 \times D_2 \to R$ defined as follows: $F(n, x) = f_n(x)$.

$\lim_n F(n, x) = \lim_n f_n(x)$ exists for each x and is (nearly) uniform in x since $\varepsilon > 0 \to \exists N \ni |f_n(x) - f(x)| < \varepsilon$ for $n \geq N$. Thus the iterated limits exist and may be interchanged; i.e.

$$f(x_0) = \lim_{n \to \infty} f_n(x_0) = \lim_{n \to \infty} \lim_{x \to x_0} f_n(x) = \lim_{x \to x_0} \lim_{n \to \infty} f_n(x) = \lim_{x \to x_0} f(x_0).$$

Hence f is continuous at x_0.

N.B. In the above example we could have obtained continuity of f at x_0 by assuming only that $f_n \to f$ nearly uniformly at x_0 (i.e with the D_2 relation above) and that each f_n is continuous at x_0.

Example 7 (See Helsel [1]). Let the sequence of r.v. functions f_n converge to f on an interval I containing x_0. Let the derivatives f'_n exist on I and converge (nearly) uniformly at x_0. Then $f'(x_0)$ exists and $f'_n(x_0) \to f'(x_0)$ on I; i.e. $\lim\limits_{n\to\infty} \lim\limits_{x\to x_0} \frac{f_n(x)-f_n(x_0)}{x-x_0} = \lim\limits_{x\to x} \lim\limits_{n\to\infty} \frac{f_n(x)-f_n(x_0)}{x-x_0}$.

Proof: Fix $x_0 \in I$ and use the directing relation on I which describes $x \to x_0$. Consider

$$F(n, x) = \frac{f_n(x)-f_n(x_0)}{x-x_0}.$$

By the mean value theorem

$$\frac{[f_m(x)-f_m(x_0)]-[f_n(x)-f_n(x_0)]}{(x-x_0)} = f'_m(\xi)-f'_n(\xi)$$

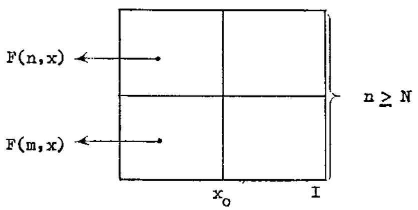
for some ξ between x and x_0;

thus

$$F(m, x)-F(n, x) = f'_m(\xi) = f'_n(\xi)$$

for some ξ between x and x_0.

But f'_n converges nearly uniformly at x so $\varepsilon > 0 \to \exists N, \delta$ such that $|f'_m(\xi) - f'_n(\xi)| < \varepsilon$ when $n, m \geq N$ *for all* $\xi \in I$ such that $|\xi - x_0| < \delta$. Thus $\varepsilon > 0 \to \exists N$ such that $|F(m, x) - F(n, x)| < \varepsilon$ when $n, m \geq N$ for all $x \in I \, |x - x_0| < \delta$, $x \neq x_0$.



This says that $F(n, x)$ is (nearly) uniformly Cauchy in x. Both single limits

$$\lim_{n\to\infty} F(n, x) = \frac{f(x)-f(x_0)}{x-x_0}$$

$$\lim_{x\to x_0} F(n, x) = f'_n(x_0)$$

exist by assumption. Thus both iterated limits exist and are equal; i.e.

$$\exists f'(x_0) = \lim_{n\to\infty} f'_n(x_0).$$

Note that we do not assume continuity of the derivatives f'_n as is usually done. Also, if we assume that $f'_n \Rightarrow$ on I and that $f_n^{(x)}$ converges *for some* x, then $f_n \Rightarrow$ to some f and $f'_n \to f'$ on I; a similar proof holds.

Example 8. Let $\{f_n\}$ be a sequence of real valued functions defined on an interval $I[a, b]$. We call

$$P = \{a = x_0 < \xi_1 < x_1 < \xi_2 < x_2 < \cdots x_{n-1} < \xi_n < x_n = b\}$$

a partition of I. If we define $P_2 \succ P_1$ to mean $\|P_2\| \leq \|P_1\|$ where $\|P\| = \max\limits_{1\leq i\leq n} |x_i - x_{i-1}|$, then the class $\mathcal{P}$ of all partitions of I is clearly directed by $\succ$.

For each $f: I \to R$, we define $S(P, f) = \sum\limits_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$. Thus

$$\int_a^b f(x)\,dx = \lim_{P\in\mathcal{P}} S(P, f)$$

if this limit exists. If $\lim_{n\to\infty} f_n(x) = f_0(x)$ for each $x \in I$, it is clear that $\lim_{n\to\infty} S(P, f_n) = S(P, f)$ for every $P \in \mathcal{P}$; if f_n converges uniformly to f_0 on I, then $\lim_{n\to\infty} S(P, f_n)$ is uniform in P. If we assume in addition that each f_n is integrable (i.e. that $\lim_{P\in\mathcal{P}} S(P, f_n)$ exists for each f_n), then the conditions of Theorem 3 are satisfied and

$$\lim_P \lim_n S(P, f_n) = \lim_n \lim_P S(P, f_n);$$

i.e.

$$\lim_P S(P, f_0) = \lim_n \int_a^b f_n(x)\, dx.$$

Thus $\int_a^b f_0(x)\, dx$ exists and $\int_a^b f_0(x)\, dx = \lim_{n\to\infty} \int_a^b f_n(x)\, dx.$

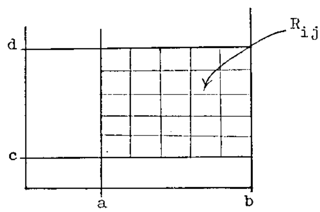
Example 9. Let $R = [a, b] \times [c, d]$ be a bounded rectangle in the plane. We ask when $\int_c^d \int_a^b f(x, y)\, dx dy = \int_a^b \int_c^d f(x, y)\, dy dx.$

Let $P = \{a = x_0 < \xi_1 < x_1 < \cdots < x_{n-1} < \xi_n < x_n = b\}$ be a partition of the interval $[a, b]$ and

$$Q = \{c = y_0 < \eta_1 < y_1 < \cdots < y_{m-1} < \eta_m < u_m = b\}$$

be a partition of the interval $[c, d]$. We form the partition

$$P \times Q = \{R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j];$$
$$(\xi_i, \eta_j) \mid i, j\} \quad \text{of} \quad [a, b] \times [c, d].$$



We shall define the norm of the partition $P \times Q = R$, $\|R\|$, as the $\max_{i,j} |R_{ij}|$ where $|R_{ij}| = \max(|P_i|, |Q_j|)$. For such partitions R_1, R_2 of $[a, b] \times [c, d]$ we define $R_1 > R_2$ to mean that each R_{ij} of R_1 is contained in some R_{ij} of R_2. If $P \times Q = R = \{R_{ij}; (\xi_i, \eta_j)\}_{ij}$ we define $S(P \times Q) = S(R) = \sum_{ij} f(\xi_i, \eta_j) A(R_{ij})$ where $A(R_{ij})$ = area of R_{ij}.

$S(P \times Q)$ is nearly uniformly Cauchy in Q if

$$\varepsilon > 0 \to \exists P_1, Q_1 \text{ such that } |S(P \times Q) - S(P^* \times Q| < \varepsilon$$

when

$$P, P^* > P_1 \text{ and } Q > Q_1.$$

Let us assume that the y-sections $f(\xi, y)$ are uniformly equicontinuous functions of x; i.e.,

$$\varepsilon > 0 \to \exists \delta > 0 \text{ such that } |f(\xi^*, y) - f(\xi^*, y)| < \frac{\varepsilon}{2(b-a)(c-d)+1}$$

for all y when $|\xi-\xi^*|<\delta$. Then if P_1 is any partition of $[a, b]$ for which $\|P_1\|<\delta$ and Q is any partition of the interval $[c, d]$, we have for $P^*\succ P_1$

$$|S(P_1, Q)-S(P^*, Q)|=\Big|\sum_{ij} f(\xi_i, \eta_j)(x_i-x_{i-1})(y_j-y_{j-1})$$

$$-\sum_{k^*, j} f(\xi_{k^*}, \eta_j)(x_{k^*}-x_{k^*-1})(y_j-y_{j-1})\Big|$$

$$=\sum_j |y_j-y_{j-1}|\,\Big|\sum_i f(\xi_i, \eta_j)(x_i-x_{i-1})-\sum_{k^*} f(\xi_{k^*}, \eta_j)(x_{k^*}-x_{k^*-1})]\Big|=$$

$$=\sum_j |y_j-y_{j-1}|\,\Big|\sum_i f(\xi_i, \eta_j)\Big[\sum_{x_{k^*-1}=x_{i-1}}^{x_{k^*}=x_i}(x_{k^*}-x_{k^*-1})\Big]-\sum_{k^*} f(\xi_{k^*}, \eta_j)(x_{k^*}-x_{k^*-1})\Big|$$

$$\leq \sum_j |y_j-y_{j-1}|\sum_i \sum_{x_{k^*-1}=x_{i-1}}^{x_{k^*}=x_i} |f(\xi_i, \eta_j)-f(\xi_k^*, \eta_j)|\,(x_k^*-x_{k^*-1})$$

$$<\sum_j |y_j-y_{j-1}|\frac{\varepsilon}{2(c-d)+1}<\frac{\varepsilon}{2}$$

Thus $|S(P, Q)-S(P^*, Q)|<\varepsilon$ when $P, P^*\succ P_1$ and Q is any partition of $[c, d]$. Now, in order to apply the interchange-of-limits theorem we need only assume that the single limits exist; i.e. that

$$\lim_P S(P\times Q) \text{ and } \lim_Q S(P\times Q)$$

exist. But

$$\lim_P S(P\times Q)=\lim_P \sum_{ij} f(\xi_1, \eta_j)(x_i-x_{i-1})(y_j-y_{j-1})$$

$$=\sum_j\Big[\lim_P \sum_i f(\xi_i, \eta_j)(x_i-x_{i-1})\Big](y_j-y_{j-1})$$

$$=\sum_j\Big[\int_a^b f(x, \eta_j)\,dx\Big](y_j-y_{j-1}).$$

Similarly

$$\lim_Q S(P\times Q)=\sum_i\Big[\int_c^d f(\xi_i, y)\,dy\Big](x_i-x_{i-1}).$$

Thus the single limits exist provided all sections (x or y) are integrable. Then, observing that

$$\lim_Q \lim_P S(P\times Q=\lim_Q \sum_j\Big[\int_a^b f(x, \eta_j)\,dx\Big](y_j-y_{j-1})$$

$$=\int_c^d\int_a^b f(x, y)\,dx\,dy$$

$$\text{and } \lim_P \lim_Q S(P\times Q)=\int_a^b\int_c^d f(x, y)\,dy\,dx$$

we apply the interchange-of-limits theorem to conclude that

$$\lim_{P, Q} S(P\times Q)=\int_a^b\int_c^d f(x, y)\,dy\,dx=\int_c^d\int_a^b f(x, y)\,dx\,dy$$

when all x-(or y,-respectively) sections are integrable and the family of y-sections (respectively x-sections) is uniformly equicontinuous. Of course

$$\lim_{P, Q} S(P \times Q) = \int_R \int f(x, y)\, dA$$

provided the double integral exists.

We could also apply the above results to double sequences $\{s_{m,n}\}$ or double series $\sum_{n,m} A_{n,m}$.

In closing we observe that if we have a class of functions f_y and define $F(x, y) = f_y(x)$, then "nearly uniform in y" might be interpreted as "nearly equicontinuous".

5. Iterated Limits

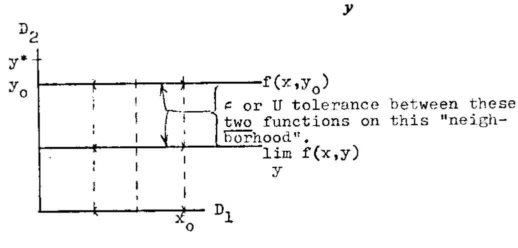
Because of the last example in which

$$\int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx$$

in the possible absence of the existence of $\int_R \int f(x, y)\, dA$, we might inquire into the nature of the interchange-of-iterated-limits in the absence of the double limit. The situation here is even more elementary. Just as in the proof that the uniform limit of continuous functions is continuous, a particular f_{n_0} which is close to f is used, a particular $f(\ , y_0)$ or $f(x_0,\ )$ which is close to the limit function will be used

Definition. If $\lim_y f(x, y)$ exists on some tail of D_1, this limit is said to be discreetly-uniform in x provided

$$U \in \mathcal{U}, y^* \in D_2 \to \exists\, y_0 \succ y^* \text{ and } \exists\, x_0 \in D_1 \text{ such that } [f(x, y_0), \lim_y f(x, y)] \in U\ \forall\, x \succ x_0$$



A similar definition is given for discreetly-uniform in y.

Theorem 4. Let f: $D_1 \times D_2 \to X$ where $(X, \mathcal{U})$ is a uniform space. Assume that the single limits $\lim_x f(x, y)$, $\lim_y f(x, y)$ exist on tails of D_2, D_1 respectively. Then the following are equivalent:

1) $\lim_y \lim_x f(x, y)$ exists and $\lim_y f(x, y)$ is discreetly-uniform in x

2) $\lim_x \lim_y f(x, y)$ exists and $\lim_x f(x, y)$ is discreetly-uniform in y

3) $\lim_y \lim_x f(x, y) = \lim_x \lim_y f(x, y)$.

Proof. It clearly suffices to show that 1) and 3) are equivalent in virtue of the symmetry of the statements.

1) $\to$ 3). $V \in \mathcal{U} \to \exists\, U = U^{-1}$ such that $\mathcal{U} \ni U^3 \subset V$.

$$\exists\, y^* \text{ such that } [\lim_x f(x, y),\ \lim_y \lim_x f(x, y)] \in U\, \forall y > y^*.$$

Then $\exists\, y_0 > y^*$ and $\exists\, x_0$ such that $[f(x, y_0),\ \lim_y f(x, y)] \in U \forall x > x_0$. But $\exists\, x_1 > x_0$ such that $[f(x, y_0),\ \lim_x f(x, y_0)] \in U \forall x > x_1$. Then

$$[\lim_y f(x, y),\ \lim_y \lim_x (x, y)] = [\lim_y f(x, y),\ f(x, y_0)] \circ [f(x, y_0),\ \lim_x f(x, x_0)] \circ$$

$$[\lim_x f(x, y_0),\ \lim_y \lim_x f(x, y)] \in U \circ U \circ U \subset V \quad \forall x > x_1.$$

Thus $\lim_x \lim_y f(x, y)$ exists and equals $\lim_y \lim_x f(x, y)$.

3) $\to$ 1). Conversely, we are assuming that both iterated limits exist and are equal. $U \in \mathcal{U} \to \exists\, V = V^{-1} \in \mathcal{U}$ such that $V^3 \subset U$. There is a $y_1 \in D_2$ such that

$$[\lim_x f(x, y),\ \lim_y \lim_x f(x, y)] \in V \quad \forall y > y_1$$

and there is an $x_1 \in D_1$ such that

$$[\lim_y f(x, y),\ \lim_x \lim_y f(x, y) \in V \quad \forall x > x_1$$

Then $y^* \in D_2 \to \exists\, y_0 > y^*,\ y_1$ and for $f(\ ,y_0)$ there is an $x_0 > x_1$ such that

$$[f(x, y_0),\ \lim_x f(x, y)] \in V \quad \forall x > x_0.$$

Thus $[f(x, y_0),\ \lim_y f(x, y)] =$

$$[f(x, y_0),\ \lim_x f(x, y_0)] \circ [\lim_x f(x, y_0),\ \lim_y \lim_x f(x, y)] \circ$$

$$[\lim_x \lim_y f(x, y),\ \lim_y f(x, y)] \in V \circ V \circ V \subset U \forall x > x_0.$$

We see that Example 3 satisfies the conditions of the above theorem while the double limit does not exist. Thus the single limits in this case are discreetly-uniform but not nearly uniform.

Corollary 4. Let $\{f_n\}$ be a convergent sequence or net of mappings from a metric space (X, d) to a uniform space $(Y, \mathcal{U})$ each of which is continuous at $x_0 \in X$. Then the limit f is continuous at x_0 if and only if for every $U \in \mathcal{U}$ and $N > 0$ there is an $n_0 > N$ and a $\delta > 0$ such that $[f_{n_0}(x), f(x)] \in U$ when $d(x_0, x) < \delta$. This is likewise equivalent to: $\lim_{x \to x_0} f(x)$ exists and for every $U \in \mathcal{U}$ and $\delta > 0$ there is an x_1 such that $d(x_0, x_1) < \delta$ and an $N > 0$ such that $[f_n(x_1), f_n(x_0)] \in U$ whenever $n > N$. [A sort of twopoint equicontinuity?].

Proof. This follows immediately from Theorem 4 when X is directed to describe convergence to x_0.

The obvious generalisation of Corollary 4 to topological spaces generalizes a result of Marjanovic [3].

BIBLIOGRAPHY

[1] R. G. Helsel, *Unpublished Real Analysis Lecture Notes*, The Ohio State University, 1963.

[2] E. J. McShane, *Partial Orderings and Moore Smith Limits*, The American Mathematical Monthly, Vol. 59, pp. 1—11. 1952.

[3] M. Marjanović, *A Note on Uniform Convergence*, Publications de l'Institut Mathématique, Tome 1 (15) pp. 109—110, 1961

[4] E. H. Moore and H. L. Smith, *A General Theory of Limits*, American Journal of Mathematics, Vol. 44, pp. 102—121, 1922.

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