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COMPLETION OF THE SPACE C_s

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In an earlier paper [8] we constructed two subspaces C_s and $\overline{C}_s$ of the field K of Mikusiński's operators [5]. C_s is a space of B_0^* type [4] and $\overline{C}_s$ of B_0 type, $C_s \subset \overline{C}_s$. Now we constructed the least space of B_0 type containing C_s.

In the field K of Mikusiński's operators we have the definition of the limit of a sequence. For the definition of sequence convergence in K it is not always true that $\overline{(\overline{A})} = \overline{A}$ [9]. Morever, for every $k \geqslant 2$ there exists a subset $B \subset K$ such that $\overline{B}^{(k)} \neq \overline{B}^{(k-1)}$ [2] (we profit by the notation $\overline{(\overline{B})} \equiv \overline{B}^{(2)}$). For this reason it is impossible to make good use of the known theory of topological spaces. T. Boehme [1] constructed a topology which has special properties in connection with the defined limit in K. Some authors such as D. O. Norris [7] and G. Krabbe [3] defined a topology for subsets of K. Our subspaces C_s and $\overline{C}_s$ were constructed to make easier a theory of differential equations in K.

First some notations.

We shall let $f = \{f(t)\}$ denote the represetation of $f(t)$ in C (C is the algebra of continuous complex valued functions defined on R^+) and s the differential operator. The elements of K are „convolution quotients“ $\frac{f}{g}$, where f and g belong to C, $g \neq 0$. Furthermore, we denote by $F_p(t) = t^{-p-1}\Phi(-p, -\sigma; -t^{-\sigma})$, $t > 0$ and $F_p(0) = 0$, $p \geqslant 0$, $0 < \sigma < 1$, where Φ is the known function of E. M. Wright [10]:

$$\Phi(\nu, \rho; z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+1)\,\Gamma(\nu + \rho n)}$$

This function $F_p(t)$ has the following properties used here:

1. $F_p \in C$, $p \geqslant 0$; **2.** $s^\beta F_p = F_{p+\beta}$, $p \geqslant 0$, $\beta \geqslant 0$.

By C_s we denoted the set of all members of K of the form $s^\beta f$, $f \in C$, $\beta \geqslant 0$. C_s forms a commutative algebra where the product, sum and scalar product are defined in the usual way, making use of those in K.

The consequence of the second property of F_p is that the convolution by F_p maps C_s into C. So we can define a sequence of seminorms $\nu_{k,p}$ in C_s:

$$\nu_{k,p}(s^\beta f) = \underset{0 \leq t \leq k}{\text{Max}} \left| \int_0^t F_{p+\beta}(t-u) f(u)\, du \right|$$

condensed to:

$$\nu_{k,p}(s^{\beta}f) = \underset{0 \le t \le k}{\text{Max}} |F_{p+\beta}f|.$$

This is a monotone and saturated sequence of seminorms.

It is easy to see that for a fixed $p \geqslant 0$ C_s is a space of B_0^* type. The topology τ_p defined in this way is finer than by Mikusiński, but not so fine as by Norris.

The space C_s is not isomorphic with a normed space and is not complete for any $p \geqslant 0$

We will now introduce the space $\overline{C}_s^p$ which is the least space of B_0 type which contains C_s.

Definition. *$\overline{C}_s^p$ is the set of all elements $\frac{v}{F_p}$, where $v \in C_0$, C_0 is the subalgebra of C of those elements f for which $f(0) = 0$.*

The sequence of seminorms $\nu_{k,p}$ for fixed p is defined in $\overline{C}_s^p$ in like manner as in C_s,

$$\nu_{k,p}\left(\frac{v}{F_p}\right) = \underset{0 \le t \le k}{\text{Max}} |v(t)|$$

The space $\overline{C}_s^p$ is homeomorphic with C and complete.

Proposition. *For a fixed p $\overline{C}_s^p$ is the least space of B_0 type containing C_s.*

Proof. — We know that for every $p \geqslant 0$, $\beta \geqslant 0$

$$s^{\beta} = \frac{F_{p+\beta}}{F_p} \text{ and } s^{\beta}f = \frac{F_{p+\beta}f}{F_p} = \frac{G}{F_p}.$$

By supposition that $f \in C$, it follows that $G(0) = 0$ and $s^{\beta}f = \frac{G}{F_p}$ belongs to $\overline{C}_s^p$. Consequently $C_s \subset \overline{C}_s^p$.

We shall show that C_s is also dense in $\overline{C}_s^p$, $p \geqslant 0$. This fact follows from Mikusiński's theorem [6]:

Theorem. *For a fixed $g \in C[0, T]$, which does not vanish identically in the right neighbourhood of 0 the set of convolutions gk with $k \in C_0^{\infty}[0, T]$ is dense in $C_0[0, T]$.*

$C_0[0, T]$, is the subclass of functions belonging to $C[0, T]$ which vanish at 0.

We shall let $\frac{v}{F_p} \in \overline{C}_s^p$. In view of Mikusiński's theorem just mentioned for a fixed p and every n natural number there exists an element $w_n \in C_0^{\infty}[0, n]$ such that:

$$\underset{0 \le t \le n}{\text{Max}} |w_n F_{p+\beta} - v| < \frac{1}{n}.$$

This implies that:

$$\nu_{k,p}\left(s^{\beta} w_n - \frac{v}{F_p}\right) \leqslant \nu_{n,p}\left(s^{\beta} w_n - \frac{v}{F_p}\right) < \frac{1}{n}$$

for all $n \geqslant k$.

A neighbourhood of $\frac{v}{F_p}$ contains a ball of the form $\nu_{q,p}\left(x-\frac{v}{F_p}\right)<\frac{1}{m}$. We will show that $s^{\beta} w_n$ belongs to this neighbourhood for $n \geqslant \operatorname{Max}(m, q)$:

$$\nu_{q,p}\left(s^{\beta} w_n-\frac{v}{F_p}\right) \leqslant \nu_{n,p}\left(s^{\beta} w_n-\frac{v}{F_p}\right)<\frac{1}{n} \leqslant \frac{1}{m}.$$

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