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ON APPROXIMATION OF CONTINUOUS AND DIFFERENTIABLE FUNCTIONS BY FOURIER—JACOBI SERIES.

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Abstract

The question of representation of a continuous and differentiable function belonging to a certain class is examined and estimates are obtained for establishing the convergence of partial sums.

1. Representation of a function, belonging to a given class, in terms of a Fourier series of polynomials is clearly a problem of cardinal significance in analysis. Suetin [4] was able to obtain an important result in this direction by using Timan's theorem [7]. This was further generalized by Prasad [2,3] and some others. In the present investigation we carry this generalization still further and consider a function having ρ continuous derivatives on $[-1, 1]$ such that $f^{(\rho)}(x) \in \text{Lip}\ \mu$, $(0<\mu<1)$ and present a few auxiliary results which, in a straightforward way, yield estimates on various differences which are of principal importance in resolving questions of convergence. In particular, we shall restrict ourselves to representing functions in terms of Fourier series of ultraspherical polynomials.

2. Let

$$S_{n,\alpha}(x) = \sum_{k=0}^{n} C_k \overline{P}_k^{(\alpha)}(x), \qquad \alpha > -1 \tag{2.1}$$

be the n^{th} partial of the Fourier—Jacobi Series of a function $f(x)$ where

$$\overline{P}_k^{(\alpha)}(x) = \frac{[(2k+2\alpha+1)\,\Gamma(k+1)\,\Gamma(k+2\alpha+1)]^{1/2}}{2^{\alpha+\frac{1}{2}}\,\Gamma(k+\alpha+1)} P_k^{(\alpha)}(x), \tag{2.2}$$

$P_k^{(\alpha)}(x) \equiv P_k^{(\alpha,\alpha)}(x)$ being the k^{th} degree Jacobi polynomial with $\beta=\alpha$

In [2] the first author has established the following:

Theorem 2.1. *If* $f(x)$ *has* ρ *continuous derivatives on* $[-1, 1]$, $f^{(\rho)}(x) \in \text{Lip}\ \mu$, $(0<\mu<1)$ *and* $0<\alpha<1/2$, *then, for* $\rho+\mu \geq 2\alpha$,

$$|f(x) - S_{n,\alpha}(x)| \leq \frac{C_1 \ln n}{n^{\rho+\mu-2\alpha}}, \quad x \in [-1,1] \tag{2.3}$$

and

$$(1-x^2)^{1/2}\,|f(x)-S_{n,\alpha}(x)|\leq \frac{C_2\ln n}{n^{\rho+\mu-2\alpha+1}},\quad x\in[-1,1].$$

In the present note we generalize Theorem 2.1 to the following:

Theorem 2.2. *If $f(x)$ has ρ continuous derivatives on $[-1,1]$ and $f^{(\rho)}(x)\in$ Lip μ, $(0<\mu<1)$ then*

(2.5) $$|f^{(r)}(x)-S^{(r)}_{n,\alpha}(x)|\leq \frac{C_1^*\ln n}{n^{\rho+\mu-2r-2\alpha}},\quad \rho\geq 2r,\ \mu>2\alpha \qquad (0<\alpha<1/2),$$

(2.6) $$(1-x^2)^{1/2}\,|f^{(r)}(x)-S^{(r)}_{n,\alpha}(x)|\leq \frac{C_2^*\ln n}{n^{\rho+\mu-2r-2\alpha+1}},\quad \rho\geq 2r,\ \mu\geq 2\alpha \qquad (0<\alpha<1/2)$$

and

(2.7) $$(1-x^2)^{\alpha/2}\,|f(x)-S_{n,\alpha}(x)|\leq \frac{C_3^*\ln n}{n^{\rho+\mu-2\alpha+1}},\quad \rho\geq 0,\ \mu\geq 2\alpha \qquad (0<\alpha<1/2)$$

uniformly in $[-1,1]$.

For the sake of convenience we introduce the following notation:

$$P_n^{(\alpha)}(x)\equiv\omega_{n,\alpha}(x)$$

3. We recall the following well-known results which will be of frequent use.

From [1] pp 324, we have for $\gamma>-1$ and $\beta>-1$,

(3.1) $$\frac{\Gamma(n+\gamma+\beta+1)}{\Gamma(n+\gamma+1)}<dn^{\beta},\ d=a \text{ positive constant.}$$

From [5] we have for $-1\leq x\leq 1$ and $0<\alpha<1$,

(3.2) $$(1-x^2)^{\alpha/2}\,|\omega_{n,\alpha}(x)|<\frac{2^{1-\alpha}}{\Gamma(\alpha)\,n^{1-\alpha}},$$

(3.3) $$(1-x^2)^{1/2}\,|\omega_{n,\alpha}(x)|<\frac{2^{1-\alpha}}{\Gamma(\alpha)\,n^{1-\alpha}}$$

and

(3.4) $$|\omega_{n,\alpha}(x)|\leq C_3\,n^{\alpha}.$$

Using (2.2) and (3.1) — (3.4) we obtain for $-1\leq x\leq 1$ and $0<\alpha<1$,

(3.5) $$(1-x^2)^{\alpha/2}\,|\bar{\omega}_{n,\alpha}(x)|\leq C_4\,n^{-\frac{1}{2}+\alpha},$$

(3.6) $$(1-x^2)^{1/2}\,|\bar{\omega}_{n,\alpha}(x)|\leq C_5\,n^{-\frac{1}{2}+\alpha},$$

and

(3.7) $$|\bar{\omega}_{n,\alpha}(x)|\leq C_6\,n^{\frac{1}{2}+\alpha}.$$

Derivatives of $\bar{\omega}_{n,\alpha}(x)$ satisfy the inequality

(3.8) $$|\bar{\omega}^{(r)}_{n,\alpha}(x)|\leq C_7\,n^{2r+\alpha+\frac{1}{2}},\quad -1\leq x\leq 1$$

which is readily derived by using (3.7) and Markov's inequality [1]. Furthermore, making use of (3.8) and result (4.8) of [8], pp 38, we have for $-1\leq x\leq 1$,

$$(3.9)\qquad (1-x^2)^{1/2}\,|\,\overline{\omega}_{n,\alpha}^{(r)}(x)\,|\leq C_8\, n^{2r+\alpha-\frac{1}{2}}.$$

4. Some lemmas.

Lemma 4.1. *For* $-1\leq x\leq 1$ *and* $0<\alpha<1$,

$$(4.1)\qquad \int_{-1}^{1}(1-t^2)^{\alpha}\left|\sum_{k=r}^{n}\overline{\omega}_{k,\alpha}(t)\,\overline{\omega}_{k,\alpha}^{(r)}(x)\right|dt\leq C_4^{*} n^{2r+2\alpha}\ln n,$$

$$(4.2)\qquad (1-x^2)^{1/2}\int_{-1}^{1}(1-t^2)^{\alpha}\left|\sum_{k=r}^{n}\overline{\omega}_{k,\alpha}(t)\,\overline{\omega}_{k,\alpha}^{(r)}(x)\right|dt\leq C_5^{*}\, n^{2r+2\alpha-1}\ln n$$

and

$$(4.3)\qquad (1-x^2)^{\alpha/2}\int_{-1}^{1}(1-t^2)^{\alpha}\left|\sum_{k=0}^{n}\overline{\omega}_{k,\alpha}(t)\,\overline{\omega}_{k,\alpha}(x)\right|dt\leq C_9\, n^{2\alpha-1}\ln n.$$

Proof. We present the proof of (4.1) and remark that (4.2) and (4.3) can be proved along the same lines. By $\Delta_n(x)$ we designate the part of $[-1,1]$ on which $|x-t|\leq\frac{1}{n}$ and by $\delta_n(x)$ the remainder of the interval. We proceed to compute now

$$(4.4)\qquad \int_{-1}^{1}(1-t^2)^{\alpha}\left|\sum_{k=r}^{n}\overline{\omega}_{k,\alpha}(t)\,\overline{\omega}_{k,\alpha}^{(r)}(x)\right|dt=\int_{\Delta_n(x)}+\int_{\delta_n(x)}=I_1+I_2.$$

Clearly

$$I_1=\int_{\Delta_n(x)}(1-t^2)^{\alpha}\left|\sum_{k=r}^{n}\overline{\omega}_{k,\alpha}(t)\,\overline{\omega}_{k,\alpha}^{(r)}(x)\right|dt$$

$$\leq\int_{\Delta_n(x)}\sum_{k=r}^{n}(1-t^2)^{\alpha/2}\,|\,\overline{\omega}_{k,\alpha}(t)\,|\,|\,\overline{\omega}_{k,\alpha}^{(r)}(x)\,|\,dt$$

and hence using (3.5) and (3.8) we have

$$(4.5)\qquad I_1\leq C_{10}\int_{\Delta_n(x)}\sum_{k=0}^{n}k^{(2r+2\alpha)}\,dt$$

$$\leq C_{11}\, n^{(2r+2\alpha+1)}\int_{\Delta_n(x)}dt\leq C_{12}\, n^{2(r+\alpha}$$

In order to estimate the integral I_2 over $\delta_n(x)$ we recall the Christoffel formula [6].

$$(4.6)\qquad \sum_{k=0}^{n}\overline{\omega}_{k,\alpha}(t)\,\overline{\omega}_{k,\alpha}(x)=\Theta_n\frac{\overline{\omega}_{n+1,\alpha}(x)\,\overline{\omega}_{n,\alpha}(t)-\overline{\omega}_{n,\alpha}(x)\,\overline{\omega}_{n+1,\alpha}(t)}{x-t}\qquad(0<\Theta_n\leq 1).$$

On differentiating r times both the sides of (4.6)

$$(4.7)\qquad \sum_{k=r}^{n} \bar{\omega}_{k,\alpha}(t)\, \bar{\omega}_{k,\alpha}^{(r)}(x) = \Theta_n \frac{\{\omega_{n+1,\alpha}^{(r)}(x)\, \bar{\omega}_{n,\alpha}(t) - \bar{\omega}_{n,\alpha}^{(r)}(x)\, \bar{\omega}_{n+1,\alpha}(t)\}}{x-t}$$

$$+\Theta_n \sum_{\nu=0}^{r-1} \frac{(-1)^{r-\nu}\, r!\, \{\bar{\omega}_{n+1,\alpha}^{(\nu)}(x)\, \bar{\omega}_{n,\alpha}(t) - \bar{\omega}_{n,\alpha}^{(\nu)}(x)\, \bar{\omega}_{n+1,\alpha}(t)\}}{\nu!\,(x-t)^{r-\nu+1}}.$$

Hence we have

$$(4.8)\qquad I_2 = \int_{\delta_n(x)} (1-t^2)^{\alpha} \left| \sum_{k=r}^{n} \bar{\omega}_{k,\alpha}(t)\, \bar{\omega}_{k,\alpha}^{(r)}(x) \right| dt$$

$$\le \int_{\delta_n(x)} (1-t^2)^{\alpha} \left| \frac{\bar{\omega}_{n+1}^{(r)}(x)\, \bar{\omega}_{n,\alpha}(t) - \bar{\omega}_{n,\alpha}^{(r)}(x)\, \bar{\omega}_{n+1,\alpha}(t)}{x-t} \right| dt$$

$$+ \int_{\delta_n(x)} (1-t^2)^{\alpha} \left| \sum_{\nu=0}^{r-1} \frac{(-1)^{r-\nu} r!\, \{\bar{\omega}_{n+1,\alpha}^{(\nu)}(x)\, \bar{\omega}_{n,\alpha}(t) - \bar{\omega}_{n,\alpha}^{(\nu)}(x)\, \bar{\omega}_{n+1,\alpha}(t)}{\nu!\quad (x-t)^{r-\nu+1}} \right| dt = u_1 + u_2.$$

Making use of (3.6) and (3.8) and bearing in mind that for $t \in \delta_n(x)$, $|x-t| > \frac{1}{n}$ we get

$$(4.9)\qquad u_1 = \int_{\delta_n(x)} (1-t^2)^{\alpha} \left| \frac{\bar{\omega}_{n+1,\alpha}^{(r)}(x)\, \bar{\omega}_{n,\alpha}(t) - \bar{\omega}_{n,\alpha}^{(r)}(x)\, \bar{\omega}_{n+1,\alpha}(t)}{x-t} \right|$$

$$\le C_{13}\, n^{2r+\alpha+\frac{1}{2}} \int_{\delta_n(x)} (1-t^2)^{\alpha/2} \Big[|\bar{\omega}_{n,\alpha}(t)| + |\bar{\omega}_{n+1,\alpha}(t)| \Big] \frac{dt}{|x-t|}$$

$$\le C_{14}\, n^{2r+2\alpha} \int_{\delta_n(x)} \frac{dt}{|x-t|} \le C_{15}\, n^{2r+2\alpha} \ln n, \qquad x \in [-1,1].$$

Next, using (3.6) and (3.8) we have for u_2

$$(4.10)\qquad u_2 = \int_{\delta_n(x)} (1-t^2)^{\alpha} \left| \sum_{\nu=0}^{r-1} \frac{(-1)^{r-\nu} r!\{\bar{\omega}_{n+1,\alpha}^{(\nu)}(x)\, \bar{\omega}_{n,\alpha}(t) - \bar{\omega}_{n,\alpha}(x)\, \bar{\omega}_{n+1,\alpha}(t)}{\nu!\quad (x-t)^{r-\nu+1}} \right| dt$$

$$\le \int_{\delta_n(x)} (1-t^2)^{\alpha/2} \sum_{\nu=0}^{r-1} \frac{r!}{\nu!} \frac{\{|\bar{\omega}_{n+1,\alpha}^{(\nu)}(x)|\,|\bar{\omega}_{n,\alpha}(t)| + |\bar{\omega}_{n,\alpha}^{(\nu)}(x)|\,|\bar{\omega}_{n+1,\alpha}(t)|\}}{|x-t|^{r-\nu+1}}\, dt$$

$$\le C_{16} \sum_{\nu=0}^{r-1} \frac{n^{2\nu+2\alpha}}{\nu!} \int_{\delta_n(n)} \frac{dt}{|x-t|^{r-\nu+1}}$$

$$\le C_{17}\, n^{2r+2\alpha-1}, \ x \in [-1,1].$$

Thus, from (4.4), (4.5) (4.8), (4.9) and (4.10) the lemma is established.

Lemma 4.2. *For* $-1\leq x\leq 1$ *and* $0<\alpha<1$, *we have*

$$(4.11)\qquad \int_{-1}^{1}(1-t^2)^{\frac{\rho+\mu}{2}+\alpha}\left|\sum_{k=r}^{n}\bar{\omega}_{k,\alpha}(t)\,\bar{\omega}_{k,\alpha}^{(r)}(x)\right|dt\leq C_{18}\,n^{2r+2\alpha}\ln n,$$

$$(4.12)\qquad (1-x^2)^{1/2}\int_{-1}^{1}(1-t^2)^{\frac{\rho+\mu}{2}+\alpha}\left|\sum_{k=r}^{n}\bar{\omega}_{k,\alpha}(t)\,\bar{\omega}_{k,\alpha}^{(r)}(x)\right|dt\leq C_{19}\,n^{2r+2\alpha-1}\ln n,$$

and

$$(4.13)\qquad (1-x^2)^{\alpha/2}\int_{-1}^{1}(1-t^2)^{\frac{\rho+\mu}{2}+\alpha}\left|\sum_{k=0}^{n}\bar{\omega}_{k,\alpha}(t)\,\bar{\omega}_{k,\alpha}(x)\right|dt\leq C_{20}\,n^{2\alpha-1}\ln n.$$

Proof: This follows immediately from lemma 4.1.

Lemma 4.3. *Let* $f^{(q)}(x)\in$ Lip μ, $(0<\mu<1)$. *Then there exists a polynomial* $Q_n(x)$ *of degree at most n such that*

$$|f(x)-Q_n(x)|\leq\frac{C_{21}}{n^{q+\mu}}\left((1-x^2)^{\frac{q+\mu}{2}}+\frac{1}{n^{q+\mu}}\right),$$

and

$$|f^{(r)}(x)-Q_n^{(r)}(x)|\leq\frac{C_{22}}{n^{q+\mu-r}}\left((1-x^2)^{\frac{q+\mu-r}{2}}+\frac{1}{n^{q+\mu-r}}\right)$$

uniformly in $[-1,1]$ *and* $r=1,2,\dots,q$.

This was already demonstrated in [3].

5. Proof of Theorem 2.2. It suffices to prove (2.5) only for the proof of (2.6) follows similar reasoning while that of (2.7) is only a minor variant of the one already contained in [2]. Obviously then,

$$|f^{(r)}(x)-S_{n,\alpha}^{(r)}(x)|=|f^{(r)}(x)-Q_n^{(r)}(x)+Q_n^{(r)}(x)-S_{n,\alpha}^{(r)}(x)|$$

$$\leq|f^{(r)}(x)-Q_n^{(r)}(x)|+\int_{-1}^{1}(1-t^2)^{\alpha}\,|Q_n(t)-f(t)|\left|\sum_{k=r}^{n}\bar{\omega}_{k,\alpha}(t)\,\bar{\omega}_{k,\alpha}^{(r)}(x)\right|dt.$$

Using lemma 4.3, we have

$$|f^{(r)}(x)-S_{n,\alpha}^{(r)}(x)|\leq\frac{C_{22}}{n^{\rho+\mu-r}}\left[(1-x^2)^{\frac{\rho+\mu-r}{2}}+\frac{1}{n^{\rho+\mu-r}}\right]$$

$$+\frac{C_{21}}{n^{\rho+\mu}}\int_{-1}^{1}\left\{(1-t^2)^{\frac{\rho+\mu}{2}+\alpha}+\frac{(1-t^2)^{\alpha}}{n^{\rho+\mu}}\right\}\left|\sum_{k=r}^{n}\bar{\omega}_{k,\alpha}(t)\,\bar{\omega}_{k,\alpha}^{(r)}(x)\right|dt$$

$$\leq\frac{C_{23}}{n^{\rho+\mu-r}}+\frac{C_{21}}{n^{\rho+\mu}}\int_{-1}^{1}(1-t^2)^{\frac{\rho+\mu}{2}+\alpha}\left|\sum_{k=r}^{n}\bar{\omega}_{k,\alpha}(t)\,\bar{\omega}_{k,\alpha}^{(r)}(x)\right|dt$$

$$+\frac{C_{21}}{n^{2(\rho+\mu)}}\int_{-1}^{1}(1-t^2)^{\alpha}\left|\sum_{k=r}^{n}\bar{\omega}_{k,\alpha}(t)\,\bar{\omega}_{k,\alpha}^{(r)}(x)\right|dt.$$

Hence, using estimates (4.1) and (4.11) we finally obtain

$$|f^{(r)}(x)-S_{n,\alpha}^{(r)}(x)|\leq\frac{C_{23}}{n^{\rho+\mu-r}}+\frac{C_{24}\,n^{2r+2\alpha}\ln n}{n^{\rho+\mu}}+\frac{C_{25}\,n^{2r+2\alpha}\ln n}{n^{2\rho+2\mu}}$$

$$\leq C_{26}\frac{\ln n}{n^{\rho+\mu-2r-2\alpha}};\ \rho\geq 2r,\ \mu>2\alpha.$$

This completes the proof of (2.5) whence the basic theorem of this investigation is established.

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