PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 12 (26), 1971. pp. 81—83

SOME NEW GENERATING FUNCTIONS FOR THE GENERALIZED LAURICELLA'S SERIES

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(Received August 13, 1971)

In the present time vast literature is devoted for establishing generating functions for different special functions. Many authors in the formal proofs involve the principle of multidimensional mathematical induction, the Laplace and inverse Laplace transforms and Taylor's theorem.

Our aim in this note is twofold: to establish some new generating functions for a generalized Lauricella's function defined by the multiple series ([1], [2]):

$$\sum_{i_1, \ldots, i_n}^{0 \cdots \infty} \frac{(\alpha)_{mi_1 + \cdots + mi_n} (\beta_1)_{i_1} \cdots (\beta_n)_{i_n}}{(\gamma_1)_{i_1} \cdots (\gamma_n)_{i_n}} \frac{x_1^{mi_1} \cdots x_n^{mi_n}}{i_1! \cdots i_n!}$$

$$= F_{A_m}(\alpha;\ \beta_1, \ldots, \beta_n;\ \gamma_1, \ldots, \gamma_n;\ x_1, \ldots, x_n), \quad |x_k| < 1, \qquad k = 1, \ldots, n,$$

and to do this with a minimum technique; i.e. involving only Taylor's theorem.

1. The following generating function can be obtained

$$(1.1) \quad \begin{aligned} &(1+t)^{-\alpha-\tilde{n}} F_{A_m}\left(\alpha+\tilde{n};\ \beta_1, \ldots, \beta_n;\ \gamma_1, \ldots, \gamma_n;\ \frac{x_1}{1+t}, \ldots, \frac{x_n}{1+t}\right) \\ &= \sum_{\nu=0}^{\infty} \frac{(-t)^\nu}{\nu!} (\alpha+\tilde{n})_\nu F_{A_m}(\alpha+\tilde{n}+\nu;\ \beta_1, \ldots, \beta_n;\ \gamma_1, \ldots, \gamma_n;\ x_1, \ldots, x_n), \end{aligned}$$

where $(c)_n = c(c+1) \cdots (c+n-1)$.

P r o o f of (1.1). Let us consider the function

$$B(s) = s^{-\alpha-\tilde{n}} F_{A_m}\left(\alpha+\tilde{n};\ \beta_1, \ldots, \beta_n;\ \gamma_1, \ldots, \gamma_n;\ \frac{x_1}{s}, \ldots, \frac{x_n}{s}\right)$$

$$(1.2) \quad = s^{-\alpha-\tilde{n}} \sum_{i_1, \ldots, i_n}^{0 \cdots \infty} \frac{(\alpha+\tilde{n})_{mi_1 + \cdots + mi_n} (\beta)_{i_1} \cdots (\beta_n)_{i_n}}{(\gamma_1)_{i_1} \cdots (\gamma_n)_{i_n}} \times$$

$$\times \frac{x_1^{mi_1} \cdots x_1^{mi_n}}{i_1! \cdots i_n!} s^{-mi_1 - \cdots - mi_n}$$

where Re $(\alpha+\tilde{n})>0$. By differentiation ν-times with respect to s we get

$$B^{(\nu)}(s)=\sum_{i_1,\ldots,i_n}^{0\cdots\infty}\frac{(\beta_1)_{i_1}\cdots(\beta_n)_{i_n}}{(\gamma_1)_{i_1}\cdots(\gamma_n)_{i_n}}\frac{x_1^{mi_1}\cdots x_n^{mi_n}}{i_1!\cdots i_n!}(\alpha+\tilde{n})_{mi_1+\cdots+mi_n}\times$$

$$\times(\alpha+\tilde{n}+mi_1+\cdots+mi_n)(\alpha+\tilde{n}+mi_1+\cdots+mi_n+1)\times$$

$$\times\cdots(\alpha+\tilde{n}+mi_1+\cdots+mi_n+\nu-1)(-1)^\nu s^{-\alpha-\tilde{n}-mi_1-\cdots-mi_n-\nu}.$$

Since

$$(\alpha+\tilde{n})_{mi_1+\cdots mi_n}(\alpha+\tilde{n}+mi_1+\cdots+mi_n)\times$$

$$\times(\alpha+\tilde{n}+mi_1+\cdots+mi_n+1)\cdots(\alpha+\tilde{n}+mi_1+\cdots+mi_n+\nu-1)=$$

$$=(\alpha+\tilde{n})_\nu(\alpha+\tilde{n}+\nu)_{mi_1+\cdots+mi_n},$$

it follows

$$B^{(\nu)}(s)=(-1)^\nu(\alpha+\tilde{n})_\nu s^{-\alpha-\tilde{n}-\nu}\times$$

(1.3)

$$\times F_{A_m}\left(\alpha+\tilde{n}+\nu;\ \beta_1,\ldots,\beta_n;\ \gamma_1,\ldots,\gamma_n;\ \frac{x_1}{s},\ldots,\frac{x_n}{s}\right).$$

By using (1.2), (1.3) and Taylor's theorem one finds

$$B(s+t)=(s+t)^{-\alpha-\tilde{n}}F_{A_m}\left(\alpha+\tilde{n};\ \beta_1,\ldots,\beta_n;\ \gamma_1,\ldots,\gamma_n;\ \frac{x_1}{s+t},\ldots,\frac{x_n}{s+t}\right)$$

$$=\sum_{\nu=0}^{\infty}\frac{t^\nu}{\nu!}B^{(\nu)}(s)=\sum_{\nu=0}^{\infty}\frac{(-t)^\nu}{\nu!}(\alpha+\tilde{n})_\nu s^{-\alpha-\tilde{n}-\nu}\times$$

$$\times F_{A_m}\left(\alpha+\tilde{n}+\nu;\ \beta_1,\ldots,\beta_n;\ \gamma_1,\ldots,\gamma_n;\ \frac{x_1}{s},\ldots,\frac{x_n}{s}\right).$$

For $s=1$ we get (1.1)

It is easy to verify that the relations (2.2) and (2.5) in [3] for Appell's functions are particular cases of (1.1). If we put

$$m=1,\ \tilde{n}=0,\ i_3=\cdots=i_n=0,\ (\gamma_1)_{i_1}(\gamma_2)_{i_2}=(c)_{i_1+i_2},$$

the relation in (1.1) reduces to

$$(1+t)^{-\alpha}F_1\left(\alpha;\ \beta_1,\ \beta_2;\ c;\ \frac{x_1}{1+t},\ \frac{x_1}{1+t}\right)=$$

$$=\sum_{\nu=0}^{\infty}\frac{(-t)^\nu}{\nu!}(\alpha)_\nu F_1(\alpha+\nu;\ \beta_1,\ \beta_2;\ c;\ x_1,\ x_2),$$

i.e. (2.2) in [3]. If we put now in (1.1):

$$m=1,\ \tilde{n}=0,\ i_3=\cdots=i_n=0,\ (\beta_1)_{i_1}(\beta_2)_{i_2}=(b)_{i_1+i_2},$$

we obtain

$$(1+t)^{-\alpha}F_4\left(\alpha;\ b;\ \gamma_1,\ \gamma_2;\ \frac{x_1}{1+t},\ \frac{x_2}{1+t}\right)=$$

$$=\sum_{\nu=0}^{\infty}\frac{(-t)^\nu}{\nu!}(\alpha)_\nu F_4(\alpha+\nu;\ b;\ \gamma_1,\ \gamma_2;\ x_1,\ x_2),$$

i.e. (2.5) in [3], etc.

2. If we construct now the function

$$\tilde{B}(s)=\left(s+\sum_{r=1}^{n}x_r\right)^{-\lambda-\tilde{n}} F_{A_m}\left(\lambda+\tilde{n};\ \beta_1,\ldots,\beta_n;\ \gamma_1,\ldots,\gamma_n;\ \frac{2x_1}{s+\sum_{r=1}^{n}x_r},\ldots,\frac{2x_n}{s+\sum_{r=1}^{n}x_r}\right),$$

and proceed as in section **1,** we obtain

$$(2.1)\qquad \left(s+t+\sum_{r=1}^{n}x_r\right)^{-\lambda-\tilde{n}} F_{A_m}\left(\lambda+\tilde{n};\ \beta_1,\ldots,\beta_n;\ \gamma_1,\ldots,\gamma_n;\ \frac{2x_1}{s+t+\sum_{r=1}^{n}x_r},\ldots,\frac{2x_n}{s+t+\sum_{r=1}^{n}x_r}\right)=$$
$$=\sum_{\nu=0}^{\infty}\frac{(-t)^{\nu}}{\nu!}(\lambda+\tilde{n})_{\nu}\left(s+\sum_{r=1}^{n}x_r\right)^{-\lambda-\tilde{n}-\nu}\times$$
$$\times F_{A_m}\left(\lambda+\tilde{n}+\nu;\ \beta_1,\ldots,\beta_n;\ \gamma_1,\ldots,\gamma_n;\ \frac{2x_1}{s+\sum_{r=1}^{n}x_r},\ldots,\frac{2x_n}{s+\sum_{r=1}^{n}x_r}\right).$$

(2.1) give us a generating function for Lauricella's function

$$F_{A_m}\left(\lambda+\tilde{n};\ \beta_1,\ldots,\beta_n;\ \gamma_1,\ldots;\gamma_n;\ \frac{2x_1}{1+\sum_{r=1}^{n}x_r},\ldots,\frac{2x_n}{1+\sum_{r=1}^{n}x_r}\right),$$

if we set $s=1$.

When

$$s=1,\ m=1,\ \beta_i=-m_i\,(i=1,\ldots,n),\ \gamma_1=\cdots=\gamma_n=2,\ \tilde{n}=n$$

(2.1) reduces to relation (2.7) in [3]. We note that the relation (2.7) in [3] is incomplete since on its left side fails the expression $(\lambda+n)_r$, i.e. our expression $(\lambda+\tilde{n})_{\nu}$ on right side of (2.1).

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