PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 12 (26), 1971, pp. 73—80

ONE POSSIBILITY OF EXPERIMENTAL TEST OF THE THEORY OF RELATIVITY BY MEANS OF REFLECTION OF LIGHT AT A MOVING MIRROR

Marko D. Leko

(Communicated June 9, 1971)

The laws of reflection of light at a moving mirror with respect to an arbitrary inertial frame of reference are different in the classical theory and the theory of relativity. In this paper we shall show that one can realize experiments which, within the limits of accuracy of measurements, would not satisfy the results of both theories and, in this way, would represent a confirmation of one of them.

First, we shall derive the relativistic law of reflection of light at a moving mirror in the form we shall use for comparison with the corresponding classical law.

In the theory of relativity, as well as in the classical theory, the reflection of light at a fixed mirror takes place so that the angle of reflection is equal to the angle of incidence, and both frequencies of the incident and the reflected rays are equal. The laws, expressed in this manner, in the classical theory are valid only with respect to the privileged frame of reference (the absolute space) relative to which the velocity of light is the same in all directions and has the value $c = 3 \cdot 10^{10}\,\mathrm{cm}\,s^{-1}$. In the theory of relativity these laws are valid with respect to any inertial frame of reference, provided the mirror is fixed to it.

The direction and frequency of the light ray are determined by the frequency vector

$$(1) \qquad f^\alpha = \left\{ \frac{\nu}{c} l^i, \frac{\nu}{c} \right\}, \qquad (\alpha = 1, 2, 3, 4;\ i = 1, 2, 3)$$

where ν is the frequency and $\vec{l} = \{l^i\}$ the unit direction 3-vector.

Let S be the inertial frame of reference with respect to which the mirror L moves with a constant velocity $\vec{v}$ perpendicular to the mirror. Let α be the angle of incidence (the angle between the incident ray and the normal to the mirror). Let $Ox^1x^2x^3$ be the Cartesian orthogonal coordinate system rigidly connected with S in such a way that the x^1-axis is perpendicular to the mirror and the x^2-axis is complanar with the incident ray and the velocity $\vec{v}$. Then (fig. 1)

(2) $$l^i = \{\cos\alpha, \sin\alpha, 0\},$$

and the frequency vector

(3) $$f^\alpha = \left\{\frac{\nu}{c}\cos\alpha, \frac{\nu}{c}\sin\alpha, 0, \frac{\nu}{c}\right\}.$$

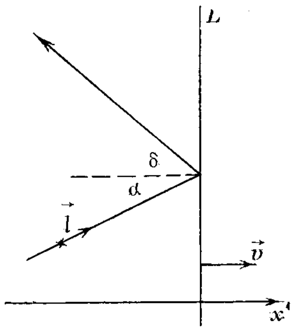

fig.1

Let the frame of reference $\bar{S}$ be that with respect to which the mirror L is at rest, and $\bar{O}\bar{x}^1\bar{x}^2\bar{x}^3$ the Cartesian orthogonal coordinate system, rigidly connected with $\bar{S}$, related to the system $Ox^1x^2x^3$ by the special Lorentz transformation

(4) $$\bar{x}^1 = \gamma(x^1 - \beta x^4), \; \bar{x}^2 = x^2, \; \bar{x}^3 = x^3, \; \bar{x}^4 = \gamma(x^4 - \beta x^1),$$

where $x^4 = ct$, t being the time, $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$.

Then one obtains the frequency vector $\bar{f}^\alpha$ of the incident ray with respect to the coordinates $\bar{x}^\alpha$ from

(5) $$\bar{f}^\alpha = f^\beta \frac{\partial \bar{x}^\alpha}{\partial x^\beta}.$$

Herefrom

(6) $$\bar{f}^\alpha = \left\{\frac{\nu}{c}\gamma(\cos\alpha - \beta), \frac{\nu}{c}\sin\alpha, 0, \frac{\nu}{c}\gamma(1 - \beta\cos\alpha)\right\},$$

and, with respect to $\bar{S}$, the frequency $\bar{\nu}$ of the incident ray is

(7) $$\bar{\nu} = \gamma(1 - \beta\cos\alpha)\nu,$$

and the unit direction vector l^i, with respect to the coordinates $\bar{x}^i$,

(8) $$\bar{l}^i = \left\{\frac{\cos\alpha - \beta}{1 - \beta\cos\alpha}, \frac{\sin\alpha}{\gamma(1 - \beta\cos\alpha)}, 0\right\}.$$

If we denote by $\bar{\alpha}$ the angle of incidence with respect to $\bar{S}$, then, according to (8),

(9) $$\cos\bar{\alpha} = \frac{\cos\alpha - \beta}{1 - \beta\cos\alpha}, \quad \sin\bar{\alpha} = \frac{\sin\alpha}{\gamma(1 - \beta\cos\alpha)}.$$

The angle of reflection $\bar{\delta}$ (the angle between the reflected ray and the normal to the mirror), with respect to $\bar{S}$, is

(10) $$\bar{\delta} = \bar{\alpha},$$

and (fig. 2) the unit direction vector $\overset{*}{\bar{l}}{}^i$ of the reflected ray with respect to the coordinates $\bar{x}^i$ is

(11) $$\overset{*}{\bar{l}}{}^i = \{-\cos\bar{\delta}, \sin\bar{\delta}, 0\} = \left\{-\frac{\cos\alpha - \beta}{1 - \beta\cos\alpha}, \frac{\sin\alpha}{\gamma(1 - \beta\cos\alpha)}, 0\right\},$$

and the frequency vector $\overset{*}{f}{}^{\alpha}$ of the reflected ray with respect to the coordinates $\bar{x}^{\alpha}$, because of $\overset{*}{\bar{\nu}}=\bar{\nu}$, considering (7) and (11), becomes

$$(12)\qquad \overset{*}{\bar{f}}{}^{\alpha}=\left\{-\gamma(\cos\alpha-\beta)\frac{\nu}{c},\ \sin\alpha\cdot\frac{\nu}{c},\ 0,\ \gamma(1-\beta\cos\alpha)\frac{\nu}{c}\right\}.$$

From

$$(13)\qquad \overset{*}{f}{}^{\alpha}=\overset{*}{\bar{f}}{}^{\beta}\frac{\partial x^{\alpha}}{\partial\bar{x}^{\beta}}$$

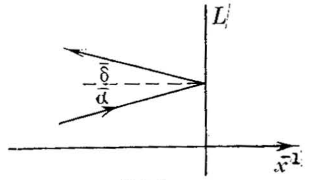

fig.2

we obtain for the frequency vector of the reflected ray with respect to the coordinates x^{α}

$$(14)\qquad \overset{*}{f}{}^{\alpha}=\left\{\gamma^2[2\beta-(1+\beta^2)\cos\alpha]\frac{\nu}{c},\ \sin\alpha\cdot\frac{\nu}{c},\ 0,\ \gamma^2(1+\beta^2-2\beta\cos\alpha)\frac{\nu}{c}\right\},$$

wherefrom the unit direction vector of that ray with respect to the coordinates x^i is

$$(15)\qquad \overset{*}{l}{}^{i}=\left\{\frac{2\beta-(1+\beta^2)\cos\alpha}{1+\beta^2-2\beta\cos\alpha},\ \frac{\sin\alpha}{\gamma^2(1+\beta^2-2\beta\cos\alpha)},\ 0\right\}.$$

Since (fig. 1)

$$(16)\qquad \overset{*}{l}{}^{i}=\{-\cos\delta,\ \sin\delta,\ 0\},$$

δ being the angle of reflection with respect to S, we obtain for δ

$$(17)\qquad \operatorname{tg}\delta=\sin\alpha\,\frac{1-\beta^2}{(1+\beta^2)\cos\alpha-2\beta},$$

which represents the wanted relativistic law of reflection of light at a moving mirror.

The classical law of reflection of light at a moving mirror with respect to a frame of reference S moving with a velocity $\vec{u}$ with respect to the privileged frame of reference (with respect to the absolute space), if the mirror moves relative to S with a constant velocity $\vec{v}$ perpendicular to the mirror and if the incident ray is complanar with the velocities $\vec{u}$ and $\vec{v}$, is, as was shown in an earlier paper[1)],

$$(18)\qquad \operatorname{tg}\delta=\frac{1}{A}\Big\{[1-(\beta+\beta_1\cos\lambda)^2]\Big[\sin\alpha\sqrt{1-\beta_1^2\sin^2(\alpha-\lambda)}-\beta_1\cos\alpha\sin(\alpha-\lambda)\Big]+$$
$$+2\beta_1(\beta+\beta_1\cos\lambda)\sin\lambda\Big[\cos\alpha\sqrt{1-\beta_1^2\sin^2(\alpha-\lambda)}+\beta_1\sin\alpha\sin(\alpha-\lambda)\Big]$$
$$-\beta_1[1+(\beta+\beta_1\cos\lambda)^2]\sin\lambda\Big\},$$

where

$$(19)\qquad A=(1+\beta^2-\beta_1^2\cos^2\lambda)\Big[\cos\alpha\sqrt{1-\beta_1^2\sin^2(\alpha-\lambda)}+\beta_1\sin\alpha\sin(\alpha-\lambda)\Big]-$$
$$-2(\beta+\beta_1\cos\lambda)+\beta_1[1+(\beta+\beta_1\cos\lambda)^2]\cos\lambda,$$

1) Publ. Inst. Math. N. s., 12 (26), 1971, pp. 67—71.

with $\beta = \frac{v}{c}$, $\beta_1 = \frac{u}{c}$, and λ being the angle between $\vec{u}$ and the normal to the mirror.

Let us note that for $\vec{u} = 0$, i.e. for $\beta_1 = 0$ (then S reduces to the absolute space), the formula (18) reduces to (17), i.e. that the relativistic law of reflection of light at a moving mirror with respect to any inertial frame of reference is the same as the corresponding classical law with respect to the absolute space.

In order to compare results given by (17) and (18) with possible experimental measurements, let us find the relativistic and classical angles of reflection in the second approximations with respect to the quantities β and β_1.

From (17), for $\beta = 0$, we get $\delta = \alpha$, and, in the second approximation, for the relativistic angle we may write that

$$\delta_r = \alpha + \beta f_1(\alpha) + \beta^2 f_2(\alpha). \tag{20}$$

Now, the left-hand side of (17), in the second approximation, is

$$\operatorname{tg}(\alpha + \beta f_1 + \beta^2 f_2) = \operatorname{tg}\alpha + \frac{1}{\cos^2\alpha}(\beta f_1 + \beta^2 f_2) + \frac{\sin\alpha}{\cos^3\alpha}(\beta f_1)^2, \tag{21}$$

and the right-hand side, in the same approximation, gives

$$\begin{aligned} \sin\alpha\cdot(1-\beta^2)\frac{1}{\cos\alpha\left[1-\left(\beta\cdot\frac{2}{\cos\alpha}-\beta^2\right)\right]} &= \\ = \operatorname{tg}\alpha\cdot(1-\beta^2)\left(1+\beta\cdot\frac{2}{\cos\alpha}-\beta^2+\beta^2\cdot\frac{4}{\cos^2\alpha}\right) &= \\ = \operatorname{tg}\alpha\cdot\left[1+\beta\cdot\frac{2}{\cos\alpha}+\beta^2\left(\frac{4}{\cos^2\alpha}-2\right)\right]. \end{aligned} \tag{22}$$

Equating expressions (21) and (22) we obtain (17) in the second approximation in the form

$$\beta\left(\frac{1}{\cos^2\alpha}f_1 - 2\frac{\sin\alpha}{\cos^2\alpha}\right) + \beta^2\left(\frac{1}{\cos^2\alpha}f_2 + \frac{\sin\alpha}{\cos^3\alpha}f_1^2 - 4\frac{\sin\alpha}{\cos^3\alpha} + 2\operatorname{tg}\alpha\right) = 0. \tag{23}$$

Since this equation is identically valid in β, for the functions $f_1(\alpha)$ and $f_2(\alpha)$ we get

$$f_1(\alpha) = 2\sin\alpha \tag{24}$$

and

$$f_2(\alpha) = \sin 2\alpha, \tag{25}$$

and (20) becomes

$$\delta_r = \alpha + \beta\cdot 2\sin\alpha + \beta^2\sin 2\alpha. \tag{26}$$

From (18), for $\beta = \beta_1 = 0$, we get $\delta = \alpha$, and, in the second approximation, the classical angle of reflection may be written in the form

$$\delta_{cl} = \alpha + \beta\varphi_1(\alpha, \lambda) + \beta_1\psi_1(\alpha, \lambda) + \beta^2\varphi_2(\alpha, \lambda) + \beta\beta_1\chi(\alpha, \lambda) + \beta_1^2\psi_2(\alpha, \lambda). \tag{27}$$

The left-hand side of (18), in the second approximation, is

(28)
$$\operatorname{tg}(\alpha+\beta\varphi_1+\beta_1\psi_1+\beta^2\varphi_2+\beta\beta_1\chi+\beta_1^2\psi_2)=$$
$$=\operatorname{tg}\alpha+\frac{1}{\cos^2\alpha}(\beta\varphi_1+\beta_1\psi_1+\beta^2\varphi_2+\beta\beta_1\chi+\beta_1^2\psi_2)+\frac{\sin\alpha}{\cos^3\alpha}(\beta^2\varphi_1^2+2\beta\beta_1\varphi_1\psi_1+\beta_1^2\psi_1^2),$$

while the right-hand side, in the second approximation, gives

(29)
$$\frac{\sin\alpha-\beta_1\sin\alpha\cos(\alpha-\lambda)-\beta^2\sin\alpha-\beta\beta_1\cdot 2\sin(\alpha-\lambda)-}{\cos\alpha\left\{1-\beta\cdot\frac{2}{\cos\alpha}-\beta_1\cos(\alpha-\lambda)+\beta^2-\beta_1^2\left[\frac{1}{2}\sin^2(\alpha-\lambda)+\cos^2\lambda\right]\right\}}-$$
$$-\frac{\beta_1^2\left[\frac{1}{2}\sin\alpha\sin^2(\alpha-\lambda)+\sin\alpha\cos^2\lambda-2\cos\alpha\sin\lambda\cos\lambda\right]}{\cos\alpha\left\{1-\beta\cdot\frac{2}{\cos\alpha}-\beta_1\cos(\alpha-\lambda)+\beta^2-\beta_1^2\left[\frac{1}{2}\sin^2(\alpha-\lambda)+\cos^2\lambda\right]\right\}}=$$
$$=\operatorname{tg}\alpha\left\{1-\beta_1\cos(\alpha-\lambda)-\beta^2-\beta\beta_1\cdot 2\frac{\sin(\alpha-\lambda)}{\sin\alpha}-\beta_1^2\left[\frac{1}{2}\sin^2(\alpha-\lambda)+\cos^2\lambda-\right.\right.$$
$$\left.\left.-2\frac{\cos\alpha\sin\lambda\cos\lambda}{\sin\alpha}\right]\right\}\left\{1+\beta\cdot\frac{2}{\cos\alpha}+\beta_1\cos(\alpha-\lambda)-\right.$$
$$-\beta^2+\beta_1^2\left[\frac{1}{2}\sin^2(\alpha-\lambda)+\cos^2\lambda\right]+\beta^2\cdot\frac{4}{\cos^2\alpha}+\beta\beta_1\cdot 4\frac{\cos(\alpha-\lambda)}{\cos\alpha}+$$
$$\left.+\beta_1^2\cos^2(\alpha-\lambda)\right\}=\operatorname{tg}\alpha+\beta\cdot 2\frac{\sin\alpha}{\cos^2\alpha}+\beta^2\cdot 2\frac{\sin\alpha(1+\sin^2\alpha)}{\cos^3\alpha}+$$
$$+\beta\beta_1\cdot 2\frac{\sin\lambda}{\cos^2\alpha}+\beta_1^2\cdot 2\sin\lambda\cos\lambda.$$

Equating the expessions (28) and (29) we obtain (18), in the second approximation, in the form

(30)
$$\beta\left(\frac{1}{\cos^2\alpha}\varphi_1-2\frac{\sin\alpha}{\cos^2\alpha}\right)+\beta_1\cdot\frac{1}{\cos^2\alpha}\psi_1+$$
$$+\beta^2\left[\frac{1}{\cos^2\alpha}\varphi_2+\frac{\sin\alpha}{\cos^3\alpha}\varphi_1^2-2\frac{\sin\alpha(1+\sin^2\alpha)}{\cos^3\alpha}\right]+$$
$$+\beta\beta_1\left(\frac{1}{\cos^2\alpha}\chi+2\frac{\sin\alpha}{\cos^3\alpha}\varphi_1\psi_1-2\frac{\sin\lambda}{\cos^2\alpha}\right)+$$
$$+\beta_1^2\left(\frac{1}{\cos^2\alpha}\psi_2+\frac{\sin\alpha}{\cos^3\alpha}\psi_1^2-2\sin\lambda\cos\lambda\right)=0.$$

From (30) we have

(31) $$\varphi_1 = 2 \sin \alpha,$$

(32) $$\psi_1 = 0,$$

(33) $$\varphi_2 = \sin 2\alpha,$$

(34) $$\chi = 2 \sin \lambda,$$

(35) $$\psi_2 = \cos^2 \alpha \sin 2\lambda,$$

and (27) becomes

(36) $$\delta_{cl} = \alpha + \beta \cdot 2 \sin \alpha + \beta^2 \cdot \sin 2\alpha + \beta\beta_1 \cdot 2 \sin \lambda + \beta_1^2 \cos^2 \alpha \sin 2\lambda.$$

First of all, we shall attempt to make use of disagreement of (26) and (36), taking the Earth as a frame of reference S and the mirror in a relative rest with respect to the Earth. Then in both formulae $\beta = 0$, and we have

(37) $$\delta_r = \alpha$$

and

(38) $$\delta_{cl} = \alpha + \beta_1^2 \cos^2 \alpha \sin 2\lambda,$$

β_1 being the quotient of velocities of the Earth and of light with respect to the absolute space.

From these formulae, in this case, it is seen that the angle of reflection differs from the angle of incidence only in the classical theory.

We shall consider the change of δ_{cl} with λ, i.e. with change of direction of motion of the frame of reference S with respect to the absolute space, and with fixed value of α. From (38) we obtain that the extreme values of δ_{cl} are

(39) $$\delta_{cl\,ext} = \alpha \pm \beta_1^2 \cos^2 \alpha.$$

We use a coordinate system Oxy rigidly connected with S in such a way that the point of origin coincides with one of incident ray's point (fig. 3), that the x-axis is perpendicular to the mirror, and that the y-axis is in the plane of the incident ray and the x-axis. Then the y-coordinate of a trace (intersection) of the reflected ray on the y-axis is

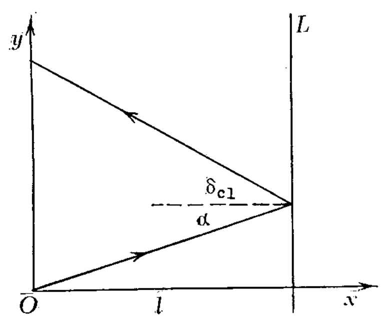

fig.3

(40) $$y_{cl} = l (\operatorname{tg} \alpha + \operatorname{tg} \delta_{cl}),$$

l being the distance of the point of origin O from the mirror. The distance s between traces which are maximally and minimally distant from the point of origin O is, then,

(41) $$s = l (\operatorname{tg} \delta_{cl\,\max} - \operatorname{tg} \delta_{cl\,\min}),$$

i.e.

(42) $$s = l \frac{\sin (\delta_{cl\,\max} - \delta_{cl\,\min})}{\cos \delta_{cl\,\max} \cdot \cos \delta_{cl\,\min}},$$

or, according to (39),

(43) $$s = l \frac{\sin (\beta_1^2 \cdot 2 \cos^2 \alpha)}{\cos (\alpha + \beta_1^2 \cos^2 \alpha) \cdot \cos (\alpha - \beta_1^2 \cos^2 \alpha)} = \frac{\sin (\beta_1^2 \cdot 2 \cos^2 \alpha)}{\cos^2 \alpha - \sin^2 (\beta_1^2 \cos^2 \alpha)}.$$

Since (38), by virtue of (36), is the second approximation of the result (18) for $\beta = 0$, (43) is justified only in that approximation, so that

$$s = 2l \frac{\beta_1^2}{1 - \beta_1^4 \cos^2 \alpha},$$

i.e.

(44) $$s = 2l\beta_1^2.$$

We note that this result is independent of α.

If the Earth is moving through the absolute space with the velocity $u = 30$ km s^{-1}, then $\beta_1 = 10^{-4}$, and, for $l = 20$ km, one obtains

(45) $$s = 0.4 \text{ mm}.$$

This displacement of the trace on the y-axis should be measurable and it has to be realized during 24 hours, i.e. during one rotation of the Earth about its own axis. Here we neglected the contribution to the velocity $\vec{u}$ which results from that rotation[2].

The same displacement (45) may be attained with a considerably smaller distance of the point of origin (the y-axis) from the mirror L. It is sufficient to place through the point of origin a second mirror, the mirror L_1, parallel to the mirror L. Then the ray reflected from L, arrives at L_1, the angle between that ray and the normal to L_1 being δ_{cl}. Taking this angle as the angle of incidence, the angle of reflection δ_1 from the mirror L_1 may be obtained from (38) substituting α by δ_{cl} and λ by $\pi - \lambda$, which is easy to show. Therefrom

(46) $$\delta_1 = \delta_{cl} - \beta_1^2 \cos^2 \delta_{cl} \sin 2\lambda,$$

or, according to (38),

(47) $$\delta_1 = \alpha + \beta_1^2 \cos^2 \alpha \sin 2\lambda - \beta_1^2 \cos^2 (\alpha + \beta_1^2 \cos^2 \alpha \sin 2\lambda) \sin 2\lambda,$$

which gives, in the second approximation,

(48) $$\delta_1 = \alpha.$$

It is seen that the trace on the y-axis after n reflections from L (fig. 4) would have the coordinate

(49) $$y_{cl} = nl\,(\text{tg}\,\alpha + \text{tg}\,\delta_{cl}),$$

and we would have

(50) $$s = 2nl\beta_1^2$$

instead of (44). For $\beta_1 = 10^{-4}$, $l = 10$ m, $n = 2000$ from (50) we have

(51) $$s = 0.4 \text{ mm}.$$

---

[2] If one would consider this contribution one would obtain u, and consequently β_1, as functions of λ. Even without detailled studies of these functions one can conclude that this will not seriously perturb the phenomenon, i.e. that even then a displacement of the trace should exist according to the classical theory. The result would, of course, be different form (45), however the difference would be small since the velocities of points of the Earth's surface due to rotation are small comparing to the velocity of 30 km s^{-1}.

Using multifold reflections from the two mirrors the change of λ may be realized simply by a rotation of the whole apparatus as a rigid body about an axis perpendicular to the xy-plane, α being constant. This makes a 24-hours experiment, using the rotation of the Earth, unnecessary.

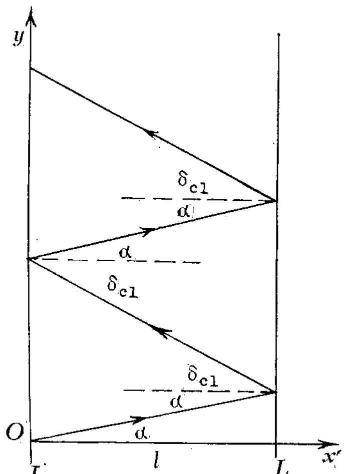

fig.4

The absence of this displacement (which could be considered any time during one year) would indicate that either the Earth is at rest relative to the absolute space or the classical theory is not valid. In order to solve this ambiguity, we may place the frame of reference (the apparatus) on a spacecraft. Then, under the assumption that the Earth is at rest relative to the absolute space, β_1 is determined by the velocity of the spacecraft relative to the Earth.

For $u = 10$ km s^{-1} (the velocity of the spacecraft relative to the Earth), we have $\beta_1 = \frac{1}{3} \cdot 10^{-4}$, and for $l = 10$ m, $n = 2000$, from (50) one obtains

(52) $$s = 0.13 \text{ mm}.$$

The absence of the displacement of the trace in the experiment on the Earth and, also, in the experiment on the spacecraft would signify that the classical theory is not valid. On the contrary, the absence of the displacement of the trace in both experiments would be, according to (37), i.e. (17), in agreement with the theory of relativity.

At the end let us note that the described experiments may be used also to check theories based on the state of motion of the light souice.