PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 12 (26), 1971, pp. 67—71

THE CLASSICAL LAW OF REFLECTION OF LIGHT AT A MOVING MIRROR WITH RESPECT TO A MOVING FRAME OF REFERENCE

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(Communicated April 21, 1971)

Let $c = 3 \cdot 10^{10}$ cm s^{-1} be the velocity of light with respect to a privileged (fixed) frame of reference (an absolute space) S_0. Let a frame of reference S move relative to S_0 with a constant velocity $\vec{u}$ and let the plane mirror L move relative to S with a constant velocity $\vec{v}$ perpendicular to the mirror plane. In this paper we shall derive, with respect to S, the classical law of reflection of light ray at the mirror L, assuming the incident ray complanar with the velocities $\vec{u}$ and $\vec{v}$.

At first, we shall derive the law of reflection at a moving mirror with respect to the absolute space (the privileged frame of reference S_0).

Let w be the component of a velocity of translatory motion of the mirror with respect to S_0 in the direction perpendicular to the mirror. (We note that a motion of the mirror in its own plane does not affect the reflection of light.) We shall consider a plane wave and two rays, l_1 and l_2, in it (fig. 1). By L_1 and L_2 we denoted positions of the mirror at the times of arrivals of the rays l_1 and l_2, respectively, at the mirror.

At the time of arrival of the ray l_1 at the point A, the ray l_2 is at B — the normal projection of A on l_2. Let C be the point of the mirror at which the ray l_2 reaches it. During the time interval Δt, necessary for l_2 to pass the distance $\overline{BC} = c\Delta t$, the mirror is displaced for $\overline{AH} = w\Delta t$, and the spherical wave, from A as a source (according to Huygens' principle), reaches the radius $r = \overline{BC} = c\Delta t$. The result of interference is the reflected plane wave with the tangent plane from C to the halfsphere of the mentioned sphere, on the side of mirror in the position L_1 wherefrom the ray is coming, as a front. Let D be the tangent point of that plane and the halfsphere. Then AD is the direction of the reflected ray. $\bar{\alpha}$ and $\bar{\delta}$ denote the angles between the incident and the reflected rays, respectively, and the normal to the mirror.

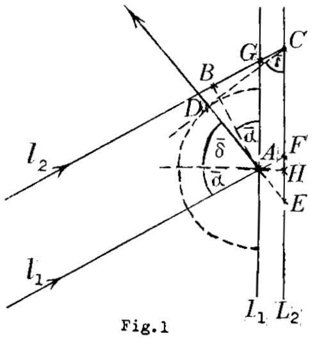

Fig. 1

Let E and F be intersections of the reflected and the incident rays, respectively, with the mirror in the position L_2 and G the intersection of the ray l_2 with the mirror in the position L_1. Then we have

$$\sin\bar{\delta}=\frac{\overline{DE}}{\overline{CE}}=\frac{\overline{AD}+\overline{AE}}{\overline{AG}+\overline{EF}}. \tag{1}$$

On account of

$$\overline{AD}=r=c\,\Delta t,$$

$$\overline{AE}=\frac{\overline{AH}}{\cos\bar{\delta}}=\frac{w\,\Delta t}{\cos\bar{\delta}},$$

$$\overline{AG}=\frac{\overline{BG}}{\sin\bar{\alpha}}=\frac{\overline{BC}-\overline{CG}}{\sin\bar{\alpha}}=\frac{c\,\Delta t-\dfrac{\overline{AH}}{\cos\bar{\alpha}}}{\sin\bar{\alpha}}=\frac{c\cos\bar{\alpha}-w}{\sin\bar{\alpha}\cos\bar{\alpha}}\,\Delta t,$$

$$\overline{EF}=\overline{FH}+\overline{EH}=\overline{AH}\,(\mathrm{tg}\,\bar{\alpha}+\mathrm{tg}\,\bar{\delta})=w\,(\mathrm{tg}\,\bar{\alpha}+\mathrm{tg}\,\bar{\delta})\,\Delta t,$$

(1) becomes

$$\sin\bar{\delta}=\frac{c+\dfrac{w}{\cos\bar{\delta}}}{\dfrac{c\cos\bar{\alpha}-w}{\sin\bar{\alpha}\cos\bar{\alpha}}+w\,(\mathrm{tg}\,\bar{\alpha}+\mathrm{tg}\,\bar{\delta})}. \tag{2}$$

The introduction of $\beta'=\dfrac{w}{c}$ gives

$$\sin\bar{\delta}\cos\bar{\delta}=\sin\bar{\alpha}\,\frac{\cos\bar{\delta}+\beta'}{1-\beta'\cos\bar{\alpha}\,(1-\mathrm{tg}\,\bar{\alpha}\,\mathrm{tg}\,\bar{\delta})}. \tag{3}$$

Expressing all the functions of $\bar{\delta}$ by $\mathrm{tg}\,\bar{\delta}$ we get

$$\frac{\mathrm{tg}\,\bar{\delta}}{\sqrt{1+\mathrm{tg}^2\,\bar{\delta}}}=\sin\bar{\alpha}\,\frac{1+\beta'\sqrt{1+\mathrm{tg}^2\,\bar{\delta}}}{1-\beta'\cos\bar{\alpha}\,(1-\mathrm{tg}\,\bar{\alpha}\,\mathrm{tg}\,\bar{\delta})}, \tag{4}$$

wherefrom

$$\begin{aligned}&\cos\bar{\alpha}\,(\cos\bar{\alpha}-2\beta'+\beta'^2\cos\bar{\alpha})\,\mathrm{tg}^2\,\bar{\delta}-\\&-2\beta'\sin\bar{\alpha}\,(1-\beta'\cos\bar{\alpha})\,\mathrm{tg}\,\bar{\delta}-(1-\beta'^2)\sin^2\bar{\alpha}=0.\end{aligned} \tag{5}$$

The solutions of this equation are

$$\mathrm{tg}\,\bar{\delta}=\sin\bar{\alpha}\,\frac{(\mp 1-\beta'^2)\cos\bar{\alpha}+\beta'\pm\beta'}{\cos\bar{\alpha}\,(\cos\bar{\alpha}-2\beta'+\beta'^2\cos\bar{\alpha})}. \tag{6}$$

The solution

$$\mathrm{tg}\,\bar{\delta}=-\mathrm{tg}\,\bar{\alpha},$$

would be valid only if the mirror does not exist (the point D from fig. 1 would be, then, on the opposite halfsphere), so that we have

$$\mathrm{tg}\,\bar{\delta}=\sin\bar{\alpha}\,\frac{1-\beta'^2}{(1+\beta'^2)\cos\bar{\alpha}-2\beta'}, \tag{7}$$

which represents the classical law of reflection of light at a moving mirror with respect to the absolute space.

Now we can proceed to he derivation of the classical law of reflection of light with respect to a frame of reference S which moves with a velocity $\vec{u}$ rela ive to S_0, at a mirror which moves with a velocity $\vec{v}$ relative to S, restricting ourselves to rays complanar with the velocities $\vec{u}$ and $\vec{v}$.

We shall rigidly connect with S the coordinate system Oxy in such a way that the point of origin O is one of incident ray's points, that the x-axis is perpendicular to the mirror, and that the y-axis is in the plane of incident ray and the velocity $\vec{u}$. Let the light ray be at O at the time $t_0 = 0$. Let $x = a$ be the equation of mirror at the time $t_0 = 0$, and let b be the ordinate of the point A at which the ray intersects the mirror.

Let t_1 be the time of arrival of the ray at the mirror, and $t = t_1 + t_2$ the time of arrival of the reflected ray at the y-axis.

By O_0, O_1, and O_2 we denoted (fig. 2) positions of the point of origin O with respect to S_0 (with respect to the absolute space) at the times $t_0 = 0$, t_1 and $t = t_1 + t_2$, respectively. The mirror L is represented in the position at the time t_1 only. (Then the ray comes at the mirror, and the point of origin has the position O_1 with respect to the absolute space.) The path of the light ray relative to the absolute space (to the system S_0) is shown by the dotted lines, and the full lines represent the correspondent path relative to the system S. Let α and δ be the angles, with respect to S, between the incident and the reflected rays, respectively, and the normal to the mirror, and $\bar{\alpha}$ and $\bar{\delta}$ the correspondent angles with respect to S_0. Our intention is to find the relation connecting α and δ.

Let us denote by λ the angle between the velocity $\vec{u}$ of the system S relative to S_0 and the x-axis. Then we have

$$\operatorname{tg}\alpha = \frac{\overline{AC}}{\overline{O_1 C}}, \quad \operatorname{tg}\bar{\alpha} = \frac{\overline{AB}}{\overline{O_0 B}},$$

i.e.,

$$\operatorname{tg}\alpha = \frac{b}{a + vt_1} \tag{8}$$

and

$$\operatorname{tg}\bar{\alpha} = \frac{b + u\sin\lambda \cdot t_1}{a + vt_1 + u\cos\lambda \cdot t_1}. \tag{9}$$

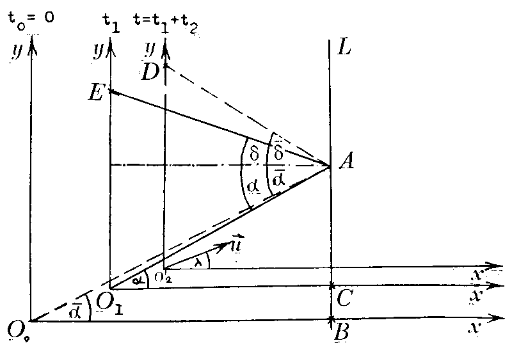

Fig.2

In order to find the relation connecting α and $\bar{\alpha}$ let us note that we also have (fig. 2)

$$\overline{O_0 B}^2 + \overline{AB}^2 = \overline{O_0 A}^2,$$

i.e.,

$$(a + vt_1 + u\cos\lambda \cdot t_1)^2 + (b + u\sin\lambda \cdot t_1)^2 = (ct_1)^2. \tag{10}$$

From (8), (9) and (10) we have to eliminate a, b and t_1.

From (8) we have

$$b = (a + vt_1)\operatorname{tg}\alpha, \tag{11}$$

and the substitution in (9) and (10) gives

$$a \operatorname{tg} \alpha + (v \operatorname{tg} \alpha + u \sin \lambda)\, t_1 = [a + (v + u \cos \lambda)\, t_1] \operatorname{tg} \bar{\alpha} \tag{12}$$

and

$$[a + (v + u \cos \lambda)\, t_1]^2 + [a \operatorname{tg} \alpha + (v \operatorname{tg} \alpha + u \sin \lambda)\, t_1]^2 = c^2 t_1{}^2. \tag{13}$$

Substituting (12) into (13) we get

$$a + (v + u \cos \lambda)\, t_1 = c t_1 \cos \bar{\alpha}. \tag{14}$$

The elimination of t_1 from (14) and (12) gives

$$\operatorname{tg} \alpha + \frac{v \operatorname{tg} \alpha + u \sin \lambda}{c \cos \bar{\alpha} - (v + u \cos \lambda)} = \frac{c \sin \bar{\alpha}}{c \cos \bar{\alpha} - (v + u \cos \lambda)}, \tag{15}$$

wherefrom, introducing the quantity $\beta_1 = \dfrac{u}{c}$,

$$\operatorname{tg} \alpha = \frac{\sin \bar{\alpha} - \beta_1 \sin \lambda}{\cos \bar{\alpha} - \beta_1 \cos \lambda}, \tag{16}$$

which represents the wanted relation connecting α and $\bar{\alpha}$.

Now, let us find the relation connecting δ and $\bar{\delta}$.

If D is the point of the y-axis at which, at the time $t = t_1 + t_2$, the ray comes after the reflection at the mirror, then this point was at the time t_1 (for which the position of mirror is just drawn) at E such that $\overline{O_1 E} = \overline{O_2 D}$. Since $\overline{AD} = c t_2$ (fig. 2) we have

$$\operatorname{tg} \delta = \frac{\overline{AD} \sin \bar{\delta} - u \sin \lambda \cdot t_2}{a + v t_1} = (c \sin \bar{\delta} - u \sin \lambda) \frac{t_2}{a + v t_1}. \tag{17}$$

From fig. 2 we also have

$$\cos \bar{\delta} = \frac{a + v t_1 - u \cos \lambda \cdot t_2}{c t_2}, \tag{18}$$

whereform

$$\frac{t_2}{a + v t_1} = \frac{1}{c \cos \bar{\delta} + u \cos \lambda}, \tag{19}$$

and (17) turns into

$$\operatorname{tg} \delta = \frac{c \sin \bar{\delta} - u \sin \lambda}{c \cos \bar{\delta} + u \cos \lambda}, \tag{20}$$

i.e.,

$$\operatorname{tg} \delta = \frac{\sin \bar{\delta} - \beta_1 \sin \lambda}{\cos \bar{\delta} + \beta_1 \cos \lambda}. \tag{21}$$

We can find the relation connecting α and δ from (16) and (21) using relation (7) connecting $\bar{\alpha}$ and $\bar{\delta}$. Introducing the quantity $\beta = \dfrac{v}{c}$, in using (7) we have to consider that $w = v + u \cos \lambda$ and $\beta' = \beta + \beta_1 \cos \lambda$, and to write (7) in the form

$$\operatorname{tg} \bar{\delta} = \sin \bar{\alpha} \frac{1 - (\beta + \beta_1 \cos \lambda)^2}{[1 + (\beta + \beta_1 \cos \lambda)^2] \cos \bar{\alpha} - 2 (\beta + \beta_1 \cos \lambda)}. \tag{22}$$

Now we have to eliminate α and δ from (16), (21) and (22). From (22) we have

$$\sin\bar{\delta} = \sin\bar{\alpha}\,\frac{1-(\beta+\beta_1\cos\lambda)^2}{1+(\beta+\beta_1\cos\lambda)^2-2(\beta+\beta_1\cos\lambda)\cos\bar{\alpha}} \tag{23}$$

and

$$\cos\bar{\delta} = \frac{[1+(\beta+\beta_1\cos\lambda)^2]\cos\bar{\alpha}-2(\beta+\beta_1\cos\lambda)}{1+(\beta+\beta_1\cos\lambda)^2-2(\beta+\beta_1\cos\lambda)\cos\bar{\alpha}}, \tag{24}$$

thus (21) becomes

$$\operatorname{tg}\delta = \frac{1}{B}\{[1-(\beta+\beta_1\cos\lambda)^2]\sin\bar{\alpha}+2\beta_1(\beta+\beta_1\cos\lambda)\sin\lambda\cos\bar{\alpha} - \beta_1[1+(\beta+\beta_1\cos\lambda)^2]\sin\lambda\}, \tag{25}$$

where

$$B = [1+(\beta+\beta_1\cos\lambda)^2-2\beta_1(\beta+\beta_1\cos\lambda)\cos\lambda]\cos\bar{\alpha}-2(\beta+\beta_1\cos\lambda)+ \beta_1[1+(\beta+\beta_1\cos\lambda)^2]\cos\lambda.$$

From (16) we get

$$\sin\bar{\alpha} = \sin\alpha\sqrt{1-\beta_1^2\sin^2(\alpha-\lambda)}-\beta_1\cos\alpha\sin(\alpha-\lambda) \tag{26}$$

and

$$\cos\bar{\alpha} = \cos\alpha\sqrt{1-\beta_1^2\sin^2(\alpha-\lambda)}+\beta_1\sin\alpha\sin(\alpha-\lambda). \tag{27}$$

Replacing (26) and (27) into (25) we obtain, finally,

$$\operatorname{tg}\delta = \frac{1}{A}\Big\{\Big[1-(\beta+\beta_1\cos\lambda)^2\Big]\Big[\sin\alpha\sqrt{1-\beta_1^2\sin^2(\alpha-\lambda)}-\beta_1\cos\alpha\sin(\alpha-\lambda)\Big] + 2\beta_1(\beta+\beta_1\cos\lambda)\sin\lambda\Big[\cos\alpha\sqrt{1-\beta_1^2\sin^2(\alpha-\lambda)}+\beta_1\sin\alpha\sin(\alpha-\lambda)\Big] - \beta_1[1+(\beta+\beta_1\cos\lambda)^2]\sin\lambda\Big\}, \tag{28}$$

with

$$A = (1+\beta^2-\beta_1^2\cos^2\lambda)\Big[\cos\alpha\sqrt{1-\beta_1^2\sin^2(\alpha-\lambda)}+\beta_1\sin\alpha\sin(\alpha-\lambda)\Big] - 2(\beta+\beta_1\cos\lambda)+\beta_1[1+(\beta+\beta_1\cos\lambda)^2]\cos\lambda, \tag{29}$$

which represents the wanted relation connecting δ and α, i.e., the classical law of reflection of light at a moving mirror with respect to a moving frame of reference.