PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 12 (26), 1971. pp. 63—66

CLAIRAUT'S EQUATIONS OF HIGHER ORDER

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(Received November 20, 1971)

1. Clairaut's differential equation

$$y-xy'+f(y')=0$$

was generalised to

$$y-xy'+\frac{1}{2!}x^2y''+\cdots+\frac{(-1)^{n-1}}{(n-1)!}x^{n-1}y^{(n-1)}+f(y^{(n)})=0 \tag{1}$$

and solved by W. H. Witty [1].

M. S. Klamkin [2] pointed out a wider generalisation given as a problem in [3]. The problem was to integrate

$$F\left(y-xy'+\frac{1}{2}x^2y'',\ y'-xy'',\ y''\right)=0, \tag{2}$$

and was noted that (2) could be generalised to differential equations of analogous form and of any order.

Equation (2) together with its general solution also appears in Kamke [4].

The generalisation pointed out by Klamkin reads:

The general solution of the differential equation

$$F(z_0,\ z_1,\ \ldots,\ z_{n-1})=0$$

where

$$z_r=y^{(r)}-xy^{(r+1)}+\frac{x^2y^{(r+2)}}{2!}-\cdots+\frac{(-1)^{n-r-1}x^{n-r-1}y^{(n-1)}}{(n-r-1)!}$$

is given by

$$y=a_0+a_1x+\frac{a_2x^2}{2!}+\cdots+\frac{a_{n-1}x^{n-1}}{(n-1)!},$$

where $a_0,\ a_1,\ \ldots,\ a_{n-1}$ are any constants satisfying $F(a_0,\ a_1,\ \ldots,\ a_{n-1})=0$.

We may note that those generalised Clairaut's equations possess solutions which are not contained in their general solutions (singular solutions).

For example, the general solution of equation

$$y=xy'-\frac{1}{2}x^2y''+y''^2$$

is

$$y = Cx + \frac{1}{2} Dx^2 + D^2 \qquad (C,\ D \text{ arbitrary constants})$$

which is a polynomial of second degree. This equation, however, is also satisfied by

$$y = \frac{1}{48} x^4,$$

which is a fourth degree polynomial.

Remark. This is not meant to be a complete bibliographical survey of Clairaut's equation. There are many more results and rediscoveries in connection with that equation, but we will not expose them here.

2. In this section we shall generalise Clairaut's equation

$$(3) \qquad u = xu_x + yu_y + f(u_x, u_y)$$

whose complete integral is given by

$$u(x, y) = Cx + Dy + f(C, D) \qquad (C,\ D \text{ arbitrary constants})$$

The first trace of this idea seems to appear in a note by D. S. Mitrinović [5] who states that the complete integral of the partial differential equation

$$(4) \qquad u = xu_x + yu_y - \frac{1}{2}(xu_x + yu_y)^{(2)} + \cdots + (-1)^{n-1} \frac{1}{n!}(xu_x + yu_y)^{(n)}$$

where

$$(xu_x + yu_y)^{(m)} = \sum_{k=0}^{m} \binom{m}{k} x^{m-k} y^k \frac{\partial^m u}{\partial x^{m-k} \partial y} k$$

is obtained by replacing each derivative in (4) by an arbitrary constant.

This result was proved in [6].

Notice that Mitrinović generalised only the "linear part" of Clairaut's equation (3).

We shall consider second order equations.

The first generalisation of (3), corresponding to (1) for $n=2$ is the equation

$$(5) \qquad u = xu_x + yu_y - \frac{1}{2}(x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy}) + f(u_{xx}, u_{xy}, u_{yy}).$$

If $u_y = 0$, this equation reduces to (1), and if $f = 0$ to (4), both for $n = 2$.

It is not difficult to see that

$$(6) \qquad u(x, y) = C_1 x + C_2 y + \frac{1}{2}(C_3 x^2 + 2C_4 xy + C_5 y^2) + f(C_3, C_4, C_5)$$

where $C_1, \ldots, C_5$ are arbitrary constants, satisfies equation (5), and that elimination of these constants from (6) and the equations

$$u_x = C_1 + C_3 x + C_4 y \qquad u_y = C_2 + C_4 x + C_5 y.$$
$$u_{xx} = C_3, \qquad u_{xy} = C_4, \qquad u_{yy} = C_5$$

yields only equation (5).

We therefore conclude that (6) is the complete integral of equation (5). A wider generalisation corresponding to (2) is the equation

$$F\left(u-xu_x-yu_y+\frac{1}{2}(x^2u_{xx}+2xyu_{xy}+y^2u_{yy}),\ u_x-xu_{xx}-yu_{xy},\ u_y-xu_{xy}-yu_{yy},\ u_{xx},\ u_{xy},\ u_{yy}\right)=0 \tag{7}$$

whose complete integral is readily seen to be

$$u(x, y)=C_0+C_1x+C_2y+\frac{1}{2}(C_3x^2+2C_4xy+C_5y^2), \tag{8}$$

where the constants $C_0, \ldots, C_5$ satisfy $F(C_0, C_1, \ldots, C_5)=0$. This means that (8) contains five arbitrary constants.

The above equations can naturally be generalised to analogous equations of higher order, or to equations for a function u depending on several variables. For example, the extension of (6) to an equation of order n is the equation

$$u=xu_x+yu_y-\frac{1}{2}(xu_x+yu_y)^{(2)}+\cdots+(-1)^{n-1}\frac{1}{n!}(xu_x+yu_y)^{(n)}$$
$$+f\left(\frac{\partial^n u}{\partial x^n}, \frac{\partial^n u}{\partial x^{n-1}\partial y}, \ldots, \frac{\partial^n u}{\partial y^n}\right)$$

whose complete integral is

$$u(x, y)=C_1^1x+C_1^2y+\frac{1}{2!}(C_2^1x^2+2C_2^2xy+C_2^3y^2)+\cdots+$$
$$+\frac{1}{n!}\left(C_n^1x^n+\binom{n}{1}C_n^2x^{n-1}y+\cdots+C_n^{n+1}y^n\right)+f(C_n^1, C_n^2, \ldots, C_n^{n+1}),$$

where C_i^k are arbitrary constants.

Similarly, the extension of (7) to the case when u depends on three variables x, y, z is the equation

$$F\left(u-xu_x-yu_y-zu_z+\frac{1}{2}(x^2u_{xx}+y^2u_{yy}+z^2u_{zz}+2xyu_{xy}+2xzu_{xz}+2yzu_{yz}),\right.$$
$$u_x-xu_{xx}-yu_{xy}-zu_{xy},\ u_y-xu_{xy}-yu_{yy}-zu_{yz},\ u_z-xu_{xz}-yu_{yz}-zu_{zz},$$
$$\left.u_{xx},\ u_{yy},\ u_{zz},\ u_{xy},\ u_{xz},\ u_{yz}\right)=0$$

whose complete integral is

$$u(x, y)=C_0+C_1x+C_2y+C_3z+\frac{1}{2}(C_4x+C_5y+C_6z+2C_7xy+2C_8xz+2C_9yz)$$

with $F(C_0, C_1, \ldots, C_9)=0$.

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