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A GENERALISATION OF THE CONCEPT OF CONVEXITY

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1. The basic definition

Definition 1. *Let $g: I^2 \to I$ be a given function such that $g(x, y) > 0$ for $y > x$ $(x, y \in I)$. We shall say that a function $f: I \to I$ is convex on I with respect to g (g-convex on I) if for all $x_1, x_2, x_3 \in I$ $(x_1 < x_2 < x_3)$ we have*

$$g(x_2, x_3)\, f(x_1) + g(x_3, x_1)\, f(x_2) + g(x_1, x_2)\, f(x_3) \geqslant 0. \tag{1}$$

This definition contains the following special cases:

Case 1. If $g(x, y) = y - x$, inequality (1) can be written in the form

$$\begin{vmatrix} 1 & x_1 & f(x_1) \\ 1 & x_2 & f(x_2) \\ 1 & x_3 & f(x_3) \end{vmatrix} \geqslant 0. \tag{2}$$

Inequality (2) is usually taken as the definition of convex functions (see, for example, **[1]** and **[2]**).

Case 2. If

$$g(x, y) = \begin{vmatrix} F(x) & G(x) \\ F(y) & G(y) \end{vmatrix}$$

inequality (1) can be represented in the form

$$\begin{vmatrix} F(x_1) & G(x_1) & f(x_1) \\ F(x_2) & G(x_2) & f(x_2) \\ F(x_3) & G(x_3) & f(x_3) \end{vmatrix} \geqslant 0. \tag{2'}$$

This definition of convexity was introduced by G. Valiron (see **[3]** and **[4]**). A special case when $F(x) = \sin \rho x$, $G(x) = \cos \rho x$ was considered by E. Phragmen and E. Lindelöf **[5]**.

Case 3. If $g(x, y) = v(y - x)$, where v is an odd function of the form

$$v(x) = x + cx^3 + o(x^3),$$

Definition 1 yields Ovčarenko's definition of convexity (see **[6]**).

2. Natural convexity

Definition 2. *We shall say that the convexity defined by inequality* (1) *is natural if for a given function g, one can find a function* $f\,(\not\equiv 0)$ *so that* (1) *reduces to equality.*

So, for instance, in Case 1 we obtain equality for $f(x)=cx$, where c is a real constant. In Case 2 convexity is natural if $f(x)=C_1 F(x)$ or $f(x)=C_2 G(x)$, where C_1, C_2 are real constants. In Case 3 convexity is natural (see [6], [7], [8]) only if $f(x)=cx$, or $f(x)=c \sin rx$, or $f(x)=c \operatorname{sh} rx$, where c and r are real constants.

The following theorem gives the necessary and sufficient conditions for the g-convexity to be natural.

Theorem 1. *In order that g-convexity is natural it is necessary and sufficient that*

$$g(x, y)=\begin{vmatrix} F(x) & F(y) \\ G(x) & G(y) \end{vmatrix}$$

where $F, G\colon I \to I$ *are arbitrary functions. Equality then occurs if and only if* $f(x)=C_1 F(x)$ *or* $f(x)=C_2 G(x)$.

Proof. In order that g-convexity is natural it is necessary and sufficient that

$$g(x_2, x_3)\, f(x_1)+g(x_3, x_1)\, f(x_2)+g(x_1, x_2)\, f(x_3)=0 \tag{3}$$

for some functions g and f.

Since $f(x)\not\equiv 0$, there exists a real number $a \in I$ such that $f(a)\neq 0$. Putting $x_3=a$ in (3) we get

$$g(x_1, x_2)=F_1(x_2)\, f(x_1)+G_1(x_1)\, f(x_2), \tag{4}$$

where

$$F_1(x)=-\frac{g(x, a)}{f(a)}, \qquad G_1(x)=-\frac{g(a, x)}{f(a)}.$$

Substituting (4) in (3), we get

$$(F_1(x_2)\, f(x_1)+G_1(x_1)\, f(x_2))\, f(x_3)+(F_1(x_3)\, f(x_2)+G_1(x_2)\, f(x_3))\, f(x_1)+$$
$$+(F_1(x_1)\, f(x_3)+G_1(x_3)\, f(x_1))\, f(x_2)=0,$$

wherefrom, setting $x_1=x_3=a$, we obtain

$$F_1(x_2)=-G_1(x_2)-\alpha f(x_2),$$

where

$$\alpha=2\, G_1(a)+2\, F_1(a).$$

Therefore, (4) becomes

$$g(x_1, x_2)=F_1(x_2)\, f(x_1)-f(x_2)\, F_1(x_1)+\alpha f(x_1)\, f(x_2). \tag{5}$$

Since (5) has to satisfy (3), we arrive at the following condition

$$\alpha f(x_1)\, f(x_2)\, f(x_3)=0$$

which for $x_1=x_2=x_3=a$, taking into account that $f(a)\neq 0$, yields

$$\alpha=0.$$

Therefore,

$$g(x_1, x) = \begin{vmatrix} f(x_1) & f(x_2) \\ F_1(x_1) & F_1(x_2) \end{vmatrix}. \tag{6}$$

Putting in (6) $f(x) = C_1 F(x)$, $F_1(x) = \frac{1}{C_1} G(x)$, $C_1 \neq 0$, we obtain

$$g(x_1, x_2) = \begin{vmatrix} F(x_1) & F(x_2) \\ G(x_1) & G(x_2) \end{vmatrix},$$

with $f(x) = C_1 F(x)$.

Similarly, setting $f(x) = C_2 G(x)$, $F_1(x) = -\frac{1}{C_2} F(x)$, we find

$$g(x_1, x_2) = \begin{vmatrix} F(x_1) & F(x_2) \\ G(x_1) & G(x_2) \end{vmatrix}, \quad \text{with} \quad f(x) = C_2 G(x).$$

This completes the proof of Theorem 1.

3. Continuity

Theorem 2. *If g is a continuous function such that $g(x, y) + g(y, x) = 0$ (i.e. $g(x, y) = G(x, y) - G(y, x)$, where G is an arbitrary function) for $x, y \in I$, than f is a continuous function on I.*

Proof. The condition of antisymmetry for g implies $g(x, x) = 0$. Let $x_2 \to x_1$ in (1). We obtain

$$f(x_1) \geqslant f(x_1 + 0).$$

Let $x_3 \to x_2$ in (1). We get

$$f(x_2) \leqslant f(x_2 + 0).$$

Hence, $f(x) = f(x + 0)$ for all $x \in I$. Similarly $f(x - 0) = f(x)$ for all $x \in I$, which completes the proof.

Remark. Notice that the condition of antisymmetry $g(x, y) + g(y, x) = 0$ is fulfilled by the functions given in Cases 1, 2, 3.

4. An inequality for g-convex functions

Theorem 3. *Let $0 \in I$, and let f be g-convex on I. Then, if $x, y > 0$, we have*

$$(g(x, x+y) + g(y, x+y)) f(0) + g(x+y, 0)(f(x) + f(y)) + (g(0, x) + g(0, y)) f(x+y) \geqslant 0. \tag{7}$$

Proof. Putting in (1) $x_1 = 0$, $x_2 = x$, $x_3 = x + y$, we get

$$g(x, x+y) f(0) + g(x+y, 0) f(x) + g(0, x) f(x+y) \geqslant 0.$$

Permutation of letters x and y gives

$$g(y, x+y) f(0) + g(x+y, 0) f(y) + g(0, y) f(x+y) \geqslant 0,$$

and addition of the last two inequalities yields (7).

Case 1. If $g(x, y)=y-x$, we obtain the following inequality

$$f(x)+f(y) \leqslant f(x+y)+f(0),$$

which was proved by M. Petrović in [9].

Case 2. For Valiron's convexity inequality (7) becomes

$$\begin{vmatrix} F(0) & G(0) \\ F(x+y) & G(x+y) \end{vmatrix} (f(x)+f(y)) \tag{8}$$

$$\leqslant \begin{vmatrix} F(0) & G(0) \\ F(x)+F(y) & G(x)+G(y) \end{vmatrix} f(x+y)+\begin{vmatrix} G(x+y) & F(x+y) \\ G(x)+G(y) & F(x)+F(y) \end{vmatrix} f(0).$$

Specially, if

$$f(x)=\limsup_{r \to +\infty} \frac{\log h(re^{ix})}{r^{\rho}} \quad (r>0,\ f \text{ is an entire function of order } \rho). \tag{9}$$

E. Lindelöf and E. Phragmen [10] have proved that f satisfies inequality (2') with $F(x)=\sin \rho x$, $G(x)=\cos \rho x$, where $x_1<x_2<x_3$ and $x_3-x_1<\frac{\pi}{\rho}$. Therefore, applying inequality (8) to the function f defined by (9), we obtain

$$(\sin \rho x+\sin \rho y)\left(\limsup_{r \to +\infty} \frac{\log f(re^{i(x+y)})}{r^{\rho}}+\limsup_{r \to +\infty} \frac{\log f(r)}{r^{\rho}}\right)$$

$$\geqslant \sin \rho(x+y)\left(\limsup_{r \to +\infty} \frac{\log f(re^{ix})}{r^{\rho}}+\limsup_{r \to +\infty} \frac{\log f(re^{iy})}{r^{\rho}}\right),$$

which holds for $x, y \in\left[0, \frac{\pi}{\rho}\right)$.

Case 3. Inequality (7) in Ovčarenko's case becomes

$$v(x+y)(f(x)+f(y)) \leqslant(v(x)+v(y))(f(x+y)+f(0)).$$

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