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ON FIXED POINTS

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Let (X, d) be a metric space. A mapping $T: X \to X$ is called a contraction mapping if there is a real number k, $0 < k < 1$, such that

(A) $$d(Tx, Ty) \leqslant kd(x, y)$$

for x, y in X.

The well-known Banach Contraction Principle states that a contraction mapping of a complete metric space X into itself has a unique fixed point. This theorem has been extensively used in the existence theorems of differential and integral equations. Recently Kannan [2] proved the following result.

Theorem: If T is a mapping of a complete metric space X into itself such that

(B) $$d(Tx, Ty) \leqslant \alpha \{d(x, Tx) + d(y, Ty)\}$$

for x, y in X and $0 < \alpha < \frac{1}{2}$, then T has a unique fixed point in X.

The aim of this paper is to prove some related theorems on fixed points.

Kannan [2] has shown that none of conditions (A) and (B) imply the other.

We prove the following proposition:

Suppose $k < \frac{1}{3}$. Then (A) implies (B).

Proof: $d(Tx, Ty) \leqslant kd(Tx, y)$

$$\leqslant k[d(x, Tx) + d(Tx, Ty) + d(Ty, y)].$$

This implies that

$$d(Tx, Ty) \leqslant \frac{k}{1-k}[d(x, Tx) + d(y, Ty)].$$

Now, $k < \frac{1}{3}$ implies $\alpha = \frac{k}{1-k} < \frac{1}{2}$, so that (B) is satisfied.

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Theorem 1: *Let X be a metric space with metrics d and δ such that $d(x, y) \leqslant \delta(x, y)$ for each pair x, y in X. Let X be complete with respect to d and let $T: X \to X$ be function in (X, d) and*

$$\delta(Tx, Ty) \leqslant \alpha \{\delta(x, Tx) + \delta(y, Ty)\}$$

for x, y in X and $0 < \alpha < \frac{1}{2}$. Then there exists a unique fixed point of T in X.

Proof: Let $x_0 \in X$ be an arbitrary point and consider the sequence of successive approximations

$$x_n = Tx_{n-1}, \qquad n = 1, 2, \ldots$$

Then $\{x_n\}$ is a Cauchy sequence with respect to δ. In fact,

$$\delta(T^n x, T^{n+1} x) \leqslant \left(\frac{\alpha}{1-\alpha}\right)^n \delta(x_1, x_0)$$

since $\delta(Tx, Ty) \leqslant \alpha \{\delta(x, Tx) + \delta(y, Ty)\}$.

Thus for $0 < \alpha < \frac{1}{2}$, $\delta(T^n x, T^{n+1} x) \to 0$ as $n \to \infty$. $\{x_n\}$ is a Cauchy sequence with respect to δ implies it is a Cauchy sequence with respect to d for

$$d(x, y) \leqslant \delta(x, y) \text{ for } x, y \text{ in } X.$$

Since X is complete with respect to d, therefore, $\{x_n\}$ converges to x in X.

Since all iterates converge to the same point, as can be seen in the following way - If $T^n x_0' \longrightarrow x_1$, $T^n x_0'' \longrightarrow x_2$, then

$$d(x_1, x_2) \leqslant d(x_1, T^n x_0') + d(T^n x_0', T^n x_0'') + d(T^n x_0'', x_2)$$

$$\leqslant d(x_1, T^n x_0') + \alpha \{d(T^{n-1} x_0', T^n x_0') + d(T^{n-1} x_0'', T^n x_0'')\} + d(T^n x_0'', x_2)$$

$$\longrightarrow 0, \text{ as } n \longrightarrow \infty.$$

So, let x be such a unique point. Then

$$d(Tx, T^{n+1} x) \leqslant \alpha \{d(x, T^n x) + d(T^n x, T^{n+1} x)\} \longrightarrow 0.$$

This implies that $T^{n+1} x \longrightarrow Tx$ and so $Tx = x$.

A similar theorem for contraction mapping in a metric space has been given by Maia [3].

Remark: In case T does not satisfy conditions of Theorem 1, but T^p (p is a positive integer) does satisfy, then T has a unique fixed point.

Proof: By Theorem 1, T^p has a unique fixed point say x_0. Then

$$T^p x_0 = x_0.$$

Now, $Tx_0 = Tx^{p+1} x_0 = T^p(Tx_0)$.

Thus Tx_0 is a fixed point of T^p. But T^p has a unique fixed point x_0 therefore,

$$Tx_0 = x_0.$$

Hence T has a unique fixed point.

Theorem 2: *If* (1) $\{T_i\}$ *is a sequence of mapping of* X *into itself with*

$$d(T_i x, T_i y) \leqslant \alpha\{d(x, T_i x) + d(y, T_i y)\}, \qquad (x, y \text{ in } X)$$

for each $i = 1, 2, \ldots$ *and each* T_i *has a fixed point* a_i; *and* (2) $\{T_i\}$ *converges pointwise to a mapping* $T: X \to X$ *with a fixed point* a; *then the sequence* $\{a_n\}$ *of fixed points converges to* a *where* a *is a fixed point of* T. *Also* T *has only one fixed point.*

Proof: Since $\{T_i\}$ converges pointwise to T, given $\varepsilon > 0$ there exists a positive integer N such that $n \geqslant N$ implies

$$d(Ta, T_n a) < \frac{\varepsilon}{1+\alpha}.$$

Thus for $n \geqslant N$,

$$\begin{aligned} d(a, a_n) &= d(Ta, T_n a_n) \\ &\leqslant d(Ta, T_n a) + d(T_n a, T_n a_n) \\ &< \frac{\varepsilon}{1+\alpha} + \alpha[d(a, T_n a) + d(a_n, T_n a_n)]. \end{aligned}$$

Since a_n is a fixed point of T_n, we get

$$\begin{aligned} d(a, a_n) &< \frac{\varepsilon}{1+\alpha} + \alpha\, d(a, T_n a) \\ &\leqslant \frac{\varepsilon}{1+\alpha} + \alpha[d(a, Ta) + d(Ta, T_n a)]. \end{aligned}$$

Again, since a is a fixed point of T, we have

$$\begin{aligned} d(a, a_n) &\leqslant \frac{\varepsilon}{1+\alpha} + \alpha\, d(Ta, T_n a) \\ &< \frac{\varepsilon}{1+\alpha} + \frac{\alpha\varepsilon}{1+\alpha} \\ &= \varepsilon. \end{aligned}$$

So that $\lim_{n\to\infty} a_n = a$.

Suppose b is another fixed point of T. Then by the above argument $\lim_{n\to\infty} a_n = b$. Hence, T has a unique fixed point.

Definition: *A* mapping $T: X \to X$ is called a non-expansive mapping if $d(Tx, Ty) \leqslant d(x, y)$ for all x, y in X. In particular, every contractive mapping $T: X \to X$ is non-expansive.

Cheney and Goldstein [1] proved the following theorem:

Let T be a map of a metric space X into itself such that

(1) $$d(Tx, Ty) \leqslant d(x, y),$$

(2) if $x \neq Tx$, then $d(Tx, T^2 x) < d(x, Tx)$,

(3) for some $x_0 \in X$, the sequence $\{T^n x_0\}$ has a subsequence $\{T^{n_i} x_0\}$ converging to x. Then the sequence $T^n x_0$ converges to x and x is a fixed point.

We prove the following result.

Theorem 3: *Let X be a metric space and T be a continuous function of X into itself. Suppose*

(1) $d(Tx,Ty) \leqslant d(x,y)$, *$x$ and y belonging to an everywhere dense subset M of X,*

(2) *if $x \neq Tx$, then $d(Tx, T^2 x) < d(x, Tx)$.*

(3) *for some $x_0 \in X$, the sequence $\{T^n x_0\}$ has a convergent sequence $\{T^{n_i} x_0\}$ converging to x, then the sequence $\{T^n x_0\}$ converges to x and x is a fixed point.*

Proof: The proof will follow from the above Theorem if we could show that (1) holds for all x, y in X.

Let $x, y \in X$, if $x \in M$ and $y \in X-M$, let $\{x_n\}$ be a sequence in M such that $x_n \to y$. Then

$$d(Tx, Ty) \leqslant d(Tx, Tx_n) + d(Tx_n, Ty)$$

$$\leqslant d(x, x_n) + d Tx_n, Ty).$$

Letting $n \to \infty$, we get

$$d(Tx, Ty) \leqslant d(x, y).$$

Now, consider the case when $x \in X-M$ and $y \in X-M$. Let $\{x_n\}$ be a sequence in M such that $x_n \to y$.

Then

$$d(Tx, Ty) \leqslant d(Tx, Tx_n) + d(Tx_n, Ty)$$

$$\leqslant d(x, x_n) + d(Tx_n, Ty)$$

form the preceding case.

Taking limit as $n \to \infty$, we get

$$d(Tx, Ty) \leqslant d(x, y).$$

Thus the theorem follows.

Remark: The referee suggested a more general result than the Theorem 2. in the following way:

Consider the mappig $T: X \longrightarrow X$ such that

$$d(Tx, Ty) \leqslant pd(x, Tx) + qd(y, Ty), \qquad 0 \leqslant p+q < 1.$$

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