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INTEGRALS INVOLVING PRODUCT OF BESSEL FUNCTIONS AND GENERALISED HYPERGEOMETRIC FUNCTIONS

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§ **1.** The object of this paper is to evaluate three integrals involving products of Bessel functions, generalised hypergeometric functions and Meijer's G-function. The results have been established by the application of a Lemma proved in § 2. The first integral generalises the results given by Bailey [2, p. 38; 45 and 3, p. 19] the second gives the generalisation of the results given by Saxena [11, p. 162] and Kalla [8, p. 168; 169]. For the definitions, properties and asymptotic behaviour of Bessel functions and generalised hypergeometric functions see [9].

The following are the results which will be helpful in our investigations that follows:

(1.1)
$$t^{\lambda}{}_p F_q\left({a_i \atop b_j}\middle| -x^2 t^2\right){}_P F_Q\left({A_I \atop B_J}\middle| -y^2 t^2\right)$$
$$=\sum_{n=0}^{\infty}\frac{(\lambda+2n)\Gamma(\lambda+n)}{Ln} J_{\lambda+2n}(2t)\, F\left({-n,\lambda+n;\ a_i;\ A_I \atop b_j;\ B_J};\ x^2, y^2\right)$$

(1.2)
$$F_c\left[\frac{1}{2}(\sigma+\lambda+\Sigma\,\vartheta_i-e),\quad \frac{1}{2}(\sigma+\lambda\,\Sigma\,\vartheta_i+e);\right.$$
$$\left.1+\mu,\ 1+\nu,\ 1+\vartheta_1,\ldots,\ 1+\vartheta_\gamma;\quad -\frac{a_2}{\alpha_2},\ -\frac{b^2}{\alpha^2},\ -\frac{\alpha_1^2}{\alpha_2},\ldots,\ \frac{\alpha_\gamma^2}{\alpha^2}\right]=$$
$$=\sum_{n=0}^{\infty}\frac{\Gamma(\lambda+n)}{Ln\cdot\Gamma(\lambda+2n)}\,\alpha^{-2n}\left\{\frac{1}{2}(\sigma+\lambda+\Sigma\,\vartheta_i\pm e)\right\}_n$$
$$\times F_4[-n,\ \lambda+n;\quad 1+\mu,\ 1+\vartheta;\quad a^2,\ b^2]$$
$$\times F_c\left[\frac{1}{2}(\sigma+\lambda+\Sigma\,\vartheta_i-e)+n,\quad \frac{1}{2}(\sigma+\lambda+\Sigma\,\vartheta_i+e)+n;\right.$$
$$\left.1+\lambda+2n,\ 1+\vartheta_1,\ldots,1+\vartheta_\gamma;\quad -\frac{1}{\alpha^2},\ -\frac{\alpha_1^2}{\alpha^2},\ldots,\ -\frac{\alpha_\gamma^2}{\alpha^2}\right],$$

valid for $R(\sigma+\lambda+\Sigma\,\vartheta_i\pm e)>0,\ R(\alpha)>\sum_{i=1}^{\gamma}|\mathrm{Im}(\alpha_i)|+|\mathrm{Im}(a)|+|\mathrm{Im}(b)|$.

(1.3)
$$\int_0^\infty x^{\sigma-1} K_e(\alpha x) \prod_{i=1}^{\gamma} [J_{\vartheta_i}(\alpha_i x)]\, dx$$

$$= \alpha^{\sigma-2}\, \alpha^{-\sigma-\Sigma\vartheta_i} \Gamma\left\{\frac{1}{2}(\sigma+\Sigma\vartheta_i+e)\right\} \Gamma\left\{\frac{1}{2}(\sigma+\Sigma\vartheta_i-e)\right\} \prod_{i=1}^{\gamma}(\alpha_i^{\vartheta_i}) \prod_{i=1}^{\gamma}\Gamma(1+\vartheta_i)^{-1}$$

$$\times F_c\left[\frac{1}{2}(\sigma+\Sigma\vartheta_i-e),\ \frac{1}{2}(\sigma+\Sigma\vartheta_i+e);\ 1+\vartheta_1,\ldots,1+\vartheta_\gamma;\ -\frac{\alpha_1^2}{\alpha^2},\ldots,-\frac{\alpha_\gamma^2}{\alpha^2}\right]$$

where $R(\sigma+\Sigma\vartheta_i\pm e)>0$, $\operatorname{Re}(\alpha)>\sum_{i=1}^{\gamma}|\operatorname{Im}(\alpha_i)|$,

$\Sigma\vartheta_i=\vartheta_1+\cdots+\vartheta_\gamma$ and $(\alpha\pm\beta)_n$ stands for $(\alpha+\beta)_n\,(\alpha-\beta)_n$ where

$$(\alpha)_n=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)},$$

(1.1), (1.2) and (1.3) have been given earlier by Srivastava [12, p. 246], Rathie [10, p. 262] and Saxena [11, p. 162] respectively.

§ **2.** Lemma:

(2.1)
$$\int_0^\infty t^{1}\, {}_pF_q\left({a_i \atop b_j}\Big|-x^2t^2\right) {}_PF_Q\left({A_I \atop B_J}\Big|-y^2t^2\right) f(t)\, dt$$

$$=\sum_{n=0}^{\infty}\frac{(\lambda+2n)\,\Gamma(\lambda+n)}{Ln} F\left[{-n,\lambda+n;\,a_i;\,A_I \atop b_j;\,B_J};\ x^2,y^2\right]\times\int_0^\infty J_{\lambda+2}(2t)\, f(t)\, dt,$$

provided that $R(\lambda+\zeta+1-2a_j-2A_J)<0)$, *for* $j=1,2,\ldots,p$, *and* $J=1,2,\ldots,p$. *and* $R(\lambda+\xi+1)>0$ *where* $f(t)=O(t^\zeta)$ *for large 't' and* $f(t)=O(t^\xi)$ *for small 't'. F is Kampé de Fériet function of two variables* [1].

Proof: Multiplying both sides of (1.1) by $f(t)$, integrating with respect to 't' from O to ∞ and interchanging the order of integration and summation we arrive at the result.

The change of order of integration and summation is justified by the following conditions [4, p. 500]:

(*i*) the series

$$\sum_{n=0}^{\infty}\frac{(\lambda+2n)\,\Gamma(\lambda+n)}{n!} J_{\lambda+2n}(2t)\, F\left[{-n,\lambda+n;\,a_i;\,A_I \atop b_j;\,B_J};\ x^2,y^2\right]$$

is uniformly convergent in $O\leqslant t\leqslant\beta$, β being arbitrary,

(*ii*) $f(t)$ is continuous function of t for all values of $t\geqslant t_0>0$,

(*iii*) the integral on the left converges absolutely this is so if $R(\lambda+\xi+1)>0$, $R(\lambda+\zeta+1)-2a_j-2A_J)<0$ for $j=1,2,\ldots,p$ and $J=1,2,\ldots,p$ where $f(t)=O(t^\xi)$ for small 't' and $f(t)=O(t^\zeta)$ for large 't'.

3. Applications

(*a*). If we take

$$f(t)=G^{\alpha\beta}_{\gamma\delta}\left(z^2t^2\,\Big|\,{\alpha_1,\ldots,\alpha_\gamma \atop \beta_1,\ldots,\beta_\delta}\right)$$

in (2.1), then on evaluating the integral on the right from [7, p. 91 (20)] we get

(3.1) $$\int_0^\infty t_p^\lambda F_q\left({a_i \atop b_j}\Big| -x^2 t^2\right) {}_PF_Q\left({A_I \atop B_J}\Big| -y^2 t^2\right) G^{\alpha\beta}_{\gamma\delta}\left(z^2 t^2 \Big| {\alpha_1, \ldots, \alpha_\gamma \atop \beta_1, \ldots, \beta_\delta}\right) dt$$

$$= \sum_{n=0}^\infty \frac{(\lambda+2n)\,\Gamma(\lambda+n)}{2 \cdot Ln} F\left({-n, \lambda+n; a_i; A_I \atop b_j; B_J}; x^2, y^2\right)$$

$$\times G^{\alpha, \beta+1}_{\gamma+2, \delta}\left(z^2 \Bigg| {\frac{1}{2}(1-\lambda-2n), \alpha_1, \ldots, \alpha_\gamma, \frac{1}{2}(1+\lambda+2n) \atop \beta_1, \ldots, \beta_\delta}\right)$$

provided that $\alpha+\beta<2(\gamma+\delta)$, $|\arg z^2|<\left(\alpha+\beta-\frac{1}{2}\gamma-\frac{1}{2}\delta\right)\pi$, $\mathrm{Re}\left(2a_i-\frac{3}{2}\right)<0$, $\mathrm{Re}(2B_j+\lambda+1)>0$ where $i=1, 2, \ldots, \beta$ and $j=1, 2, \ldots, \alpha$.

Particular cases of (3.1): (*i*) If we take $\alpha=1$, $\beta=\gamma=0$, $\delta=2$, $\beta_1=\frac{1}{2}\rho$, $\beta_2=-\frac{1}{2}\rho$, then (3.1) gives

(3.2) $$\int_0^\infty t^\lambda {}_pF_q\left({a_i \atop b_j}\Big| -x^2 t^2\right) {}_PF_Q\left({A_I \atop B_J}\Big| -y^2 t^2\right) J_\rho(zt)\, dt$$

$$= \sum_{n=0}^\infty \frac{(\lambda+2n)\,\Gamma(\lambda+n)}{Ln} F\left({-n, \lambda+n; a_i; A_I \atop b_j; B_J}; x^2, y^2\right) \times$$

$$\times G^{11}_{22}\left(\frac{z^2}{4} \Bigg| {\frac{1}{2}(1-\lambda-2n), \frac{1}{2}(1+\lambda+2n) \atop \frac{1}{2}\rho, -\frac{1}{2}\rho}\right).$$

where $R(1+\lambda)>0$, $R(z)>0$.

(*ii*) On taking $\alpha=\delta=2$, $\beta=\gamma=0$, $\beta_1=\frac{1}{2}\rho$, $\beta_2=-\frac{1}{2}\rho$ then (3.1) reduces to

(3.3) $$\int_0^\infty t^\lambda L_\rho(zt)\, {}_pF_q\left({a_i \atop b_j}\Big| -x^2 t^2\right) {}_PF_Q\left({A_I \atop B_J}\Big| -y^2 t^2\right) dt$$

$$= \sum \frac{(\lambda+2n)\,\Gamma(\lambda+n)}{4\,Ln} F\left({-n, \lambda+n; a_i; A_I \atop b_j; B_J}; x^2, y^2\right) \times$$

$$\times G^{21}_{22}\left(\frac{z^2}{4} \Bigg| {\frac{1}{2}(1-\lambda-2n), \frac{1}{2}(1+\lambda+2n) \atop \frac{1}{2}\rho, -\frac{1}{2}\rho}\right)$$

where $R(1+\lambda\pm\rho)>0$, $R(z)>0$.

If we set $p=P=0$, $q=Q=1$, $b_1=1+\mu$, $B_1=1+\nu$ and replace x, y and λ by $\frac{x}{2}$, $\frac{y}{2}$ and $\lambda+\mu+\nu-1$ respectively the result (3.3) gives a known result [7, p. 373 (8)] by virtue of the results (1.2) [5, p. 208 (5)] and [5, p. 209 (9)].

Also when $\rho=\pm\frac{1}{2}$ then by virtue of

$$K_{\pm\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}} \cdot e^{-x}$$

(3.3) reduces to

(3.4) $$\int_0^\infty t^{\lambda-\frac{1}{2}} e^{-zt} {}_pF_q\left({a_i \atop b_j}\middle|-x^2t^2\right) {}_PF_Q\left({A_I \atop B_J}\middle|-y^2t^2\right) dt$$

$$=\sqrt{\frac{2z}{\pi}} \sum_{n=0}^\infty \frac{(\lambda+2n)\,\Gamma(\lambda+n)}{4\,Ln} F\left({-n,\,\lambda+n;\,a_i;\,A_I \atop b_j;\,B_J};\quad x^2, y^2\right) \times$$

$$\times G_{22}^{21}\left(\frac{z^2}{4}\middle|{\frac{1}{2}(1-\lambda-2n),\ \frac{1}{2}(1+\lambda+2n) \atop \frac{1}{2}\rho,\ -\frac{1}{2}\rho}\right).$$

for $R\left(\lambda+\dfrac{1}{2}\right)>0,\ R(z)>0.$

(*iii*) Similarly when $\alpha=\delta=2,\ \beta=0,\ \gamma=1,\ \alpha_1=1-K,\ \beta_1=\dfrac{1}{2}+m,$ $\beta_2=\dfrac{1}{2}-m$, we get

(3.5) $$\int_0^\infty t^{\frac{1}{2}(\lambda-1)} e^{-\frac{1}{2}z^2t} W_{k,m}(z^2t)\, {}_pF_q\left({a_i \atop b_j}\middle|-x^2t^2\right) {}_PF_Q\left({A_I \atop B_J}\middle|-y^2t^2\right) dt$$

$$=\sum_{n=0}^\infty \frac{(\lambda+2n)\,\Gamma(\lambda+n)}{Ln} F\left({-n,\,\lambda+n;\,a_i;\,A_I \atop b_j;\,B_J};\quad x^2, y^2\right)$$

$$\times G_{32}^{21}\left(z^2\middle|{\frac{1}{2}(1-\lambda-2n),\ 1-k,\ \frac{1}{2}(1+\lambda+2n) \atop \frac{1}{2}+m,\ \frac{1}{2}-m}\right)$$

where $R\left(1+\dfrac{1}{2}\lambda\pm m\right)>0,\ R(z^2)>0.$

(b) If we take

$$f(t)=K_\rho(\alpha t)\prod_{i=1}^m [J_{\nu_i}(\alpha_i t)]$$

in (2.1) of § 2. and evaluate the integral on the right by means of (1.3) it is found that

(3.6) $$\int_0^\infty t^\lambda\, {}_pF_q\left({a_i \atop b_j}\middle|-x^2t^2\right) {}_PF_Q\left({A_I \atop B_J}\middle|-y^2t^2\right) K_\rho(\alpha t) \prod_{i=1}^m [J_{\nu_i}(\alpha_i t)]\, dt$$

$$=2^{\lambda-1}\alpha^{-\lambda-\Sigma\nu_i-1}\prod_{i=1}^m [\Gamma(1+\nu_i)]^{-1}$$

$$\times \sum_{n=0}^\infty \frac{\Gamma(\lambda+n)\,\Gamma\left\{\frac{1}{2}(1+\lambda+\Sigma\nu_i\pm\rho)+n\right\}}{Ln\cdot\Gamma(\lambda+2n)} F\left({-n,\,\lambda+n;\,a_i;\,A_I \atop b_j;\,B_J};\quad x^2, y^2\right) \times$$

$$\times F_c\left[\frac{1}{2}(1+\lambda+\Sigma\nu_i-\rho)+n,\ \frac{1}{2}(1+\lambda+\Sigma\nu_i+\rho)+n;\ 1+\lambda+2n,\ 1+\nu_1,\ldots,1+\nu_m;\right.$$
$$\left.-\frac{4}{\alpha^2},\ \frac{\alpha_1^2}{\alpha^2},\ldots,\ -\frac{\alpha_m^2}{\alpha^2}\right],$$

where $\mathrm{Re}(1+\lambda+\Sigma\nu_i\pm\rho)>0$, $\mathrm{Re}(\alpha)>2+\sum_{i=1}^{m}|I_m(\alpha_i)|$, $\Sigma\nu_i=\nu_1+\nu_2+\cdots+\nu_m$ and $\Gamma(\alpha\pm\beta)$ stands for $\Gamma(\alpha+\beta)\cdot\Gamma(\alpha-\beta)$.

As $x\to 0$, (3.6) reduces to the one given by Kalla [8, p. 168 (2.2)].

On the other hand if $p=P=0$, $q=Q=1$, $b_1=1+\mu$, $B_1=1+\nu$, (3.6) yields Saxena's formula [11, p. 162].

(c) Lastly if we take

$$f(t)=exp\left(-\frac{\alpha t^2}{4}\right)\prod_{i=1}^{m}[J_{\nu_i}(\alpha_i t)]$$

in (2.1) and evaluate the integral on the right from [6, p. 187 (43)] we see that

$$\text{(3.7)}\qquad \int_0^\infty t^\lambda\, {}_pF_q\left({a_i \atop b_j}\Big|-x^2t^2\right){}_PF_Q\left({AI \atop BJ}\Big|-y^2t^2\right)exp\left(-\frac{\alpha t^2}{4}\right)\prod_{i=1}^{m}[J_{\nu_i}(\alpha_i t)]\,dt$$

$$=2^\lambda\alpha^{-\frac{1}{2}(1+\lambda+\Sigma\nu_i)}\prod_{i=1}^{m}(\alpha_i^{\nu_i})\prod_{i=1}^{m}[\Gamma(1+\nu_i)]^{-1}$$

$$\times\sum_{n=0}^{\infty}\frac{\Gamma(\lambda+n)\,\Gamma\left\{\frac{1}{2}(1+\lambda+\Sigma\nu_i)+n\right\}}{Ln\cdot\Gamma(\lambda+2n)}\left(\frac{4}{\alpha}\right)^n F\left({-n,\,\lambda+n;\,a_i;\,AI \atop b_j;\,BJ};\quad x^2,y^2\right)$$

$$\times\Psi_2\left[\frac{1}{2}(1+\lambda+\Sigma\nu_i)+n;\quad 1+\lambda+2n,\ 1+\nu_1,\ldots,1+\nu_m;\right.$$

$$\left.-\frac{4}{\alpha},\ \frac{\alpha_1^2}{\alpha},\ldots,\ -\frac{\alpha_m^2}{\alpha}\right],$$

where $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(1+\lambda+\Sigma\nu_i)>0$ and $\Sigma\nu_i=\nu_1+\nu_2+\cdots+\nu_m$.

As $x\to 0$, (3.7) reduces to another result given by Kalla [8, p. 169 eq. (2.4)].

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