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ON FRACTIONAL INTEGRATION

S. L. Bora and R. K. Saxena

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§ 1. Introduction

The object of this paper is to investigate the relationship existing between Riemann-Liouville (fractional), Weyl (fractional integrals, Hankel and Meijer's transforms. The results have been given in the form of some theorems. The theorems have been illustrated by means of some suitable examples so as to give the images of confluent hypergeometric functions Ξ_2 and Φ_3 in Meijer Transform.. The results established here are General and include as particular cases well known results. We call

$$(1.1)\qquad g(p;\mu)=R_\mu\{f(t);p\}=\frac{1}{\Gamma(\mu)}\int_0^p f(t)\,(p-t)^{\mu-1}\,dt,$$

the Riemann-Liouville (fractional) integral of the order u and

$$(1.2)\qquad h(p;\mu)=W_\mu\{f(t);p\}=\frac{1}{\Gamma(\mu)}\int_p^\infty f(t)\,(t-p)^{\mu-1}\,dt$$

the Weyl (fractional) integral of order μ of $f(t)$.

The transforms which will be required in investigations later on are as follows: The classical Laplace transform

$$(1.3)\qquad L\{f(t);p\}=p\int_0^\infty e^{-pt}f(t)\,dt,$$

was generalised by Meijer [3, *p.* 599] in the form

$$(1.4)\qquad K_\nu\{f(t);p\}=\sqrt{\frac{2}{\pi}}\,p\int_0^\infty (pt)^{\frac{1}{2}}K_\nu(pt)\,f(t)\,dt.$$

When $\nu=\pm\frac{1}{2}$, (1.4) reduces to (1.3) by virtue of the well known identity.

$$K_{\pm\frac{1}{2}}(z)=\sqrt{\frac{\pi}{2}}\,e^{-z}.$$

The well known Hankel transform is represented by the integral equation

$$(1.5)\qquad J_\nu\{f(t);p\}=p\int_0^\infty (pt)^{\frac{1}{2}} J_\nu(pt)f(t)\,dt.$$

We shall also use the following property of fractional integrals [2, p. 182]

$$(1.6)\qquad \int_0^\infty f_1(t)\,g(t;\mu)\,dt=\int_0^\infty f_2(t)\,h(t;\mu)\,dt,$$

where

$$W_\mu\{f_1(t);p\}=h(p;\mu)\quad\text{and}\quad R_\mu\{f_2(t);p\}=g(p;\mu)$$

§ **2.** In this section we have established connections between Riemann--Liouville (fractional) integral Hankel and Meijer transforms. This has been done in the form of two theorems. By the application of a theorem the image of Ξ_2 in the Meijer transform has also been obtained.

Theorem I. *If*

$$(2.1)\qquad R_\mu\{f(t);p\}=g(p;\mu)$$

then

$$(2.2)\qquad L\{g(t^2;\mu);p\}=2^\mu p^{-\mu} K_{\mu-\frac{1}{2}}\{t^\mu f(t^2);p\},$$

provided that Riemann-Liouville integral of $|f(t)|$ *exist and* $\operatorname{Re}\mu>0$, $\operatorname{Re}(ap^{\frac{1}{2}})>0$.

Proof: We have [2, p. 203 (17)],

$$(2.3)\qquad W_\mu\{t^{-\frac{1}{1}}e^{-at^{\frac{1}{2}}};p\}=2^{\mu+\frac{1}{2}}\pi^{-\frac{1}{2}}a^{\frac{1}{2}-\mu}p^{\frac{1}{2}\mu-\frac{1}{4}}K_{\mu-\frac{1}{2}}(ap^{\frac{1}{2}})$$

where $\operatorname{Re}(\mu)>0$ and $\operatorname{Re}(ap^{\frac{1}{2}})>0$.

Using the relations (2.1) and (2.3) in (1.6) we obtain (2.2) after a little simplification.

Example: If we start with

$$f(t)=t^\alpha\,\Xi_2(\alpha+\mu,\beta,\gamma;a,bt)$$

then [4, p. 1007]

$$(2.4)\qquad R_\mu\{f(t);p\}=\frac{\Gamma(\alpha)}{\Gamma(\alpha+\mu)}p^{\alpha+\mu-1}\,\Xi_2(\alpha,\beta,\gamma;a,bp)=g(p;\mu)$$

for $\operatorname{Re}\alpha>0$, $\operatorname{Re}(\mu)>0$ and $|bp|<1$.

Putting these values of $f(t)$ and $g(p;\mu)$ in (2.2) and using [1, p. 223] we obtain

$$(2.5)\qquad K_{\mu-\frac{1}{2}}\{t^{2\alpha+\mu-2}\,\Xi_2(\alpha,\mu,\beta,\gamma;a,bt^2);p\}$$
$$=2^{-\mu}p^{1-\mu-2\alpha}\Gamma(\alpha)\Gamma(2\alpha+2\mu-1)\{\Gamma(\alpha+\mu)\}^{-1}\times$$
$$\times F_3\left(\alpha,\alpha+\mu-\frac{1}{2},\beta,\alpha+\mu,\gamma;a,\frac{4b}{p^2}\right),$$

for $\operatorname{Re}\alpha>0$, $\operatorname{Re}(\mu)>0$, $|bp|<1$.

When $\mu=1$, (2.5) gives a known result [1, p. 223].

Theorem II. *If*

(2.6) $$R_\mu\{f(t); p\}=g(p; \mu)$$

then

(2.7) $$J_\nu\{t^{\frac{1}{2}-\nu}\, g(t^2; \mu); p\}=2^\mu p^{-\mu} J_{\nu-\mu}\{t^{\mu-\nu+\frac{1}{2}} f(t^2); p\}$$

provided that Riemann-Liouville integral of $|f(t)|$ *exist*

$$p>0 \quad \text{and} \quad 0<\operatorname{Re}\mu<\frac{1}{2}\operatorname{Re}\nu+\frac{3}{4}.$$

Proof: We have [2, p. 205 (34)]

(2.8) $$W_\mu\{t^{-\frac{1}{2}\nu} J_\nu(at^{\frac{1}{2}}); p\}=2^\mu a^{-\mu} p^{\frac{1}{2}\mu-\frac{1}{2}\nu} J_{\nu-\mu}(ap^{\frac{1}{2}}),$$

where

$$a>0 \quad \text{and} \quad 0<\operatorname{Re}\mu<\frac{1}{2}\nu+\frac{3}{4}.$$

Using (2.6) and (2.8) in (1.6) we obtain (2.7) after a little simplification.

§ 3. In this section we have established connections between Weyl (fractional) integral, Hankel and Meijer transforms. This has been done in the form of two theorems. By the application of a theorem the image of Φ_3 in Meijer transform has also been obtained.

Theorem I. *If*

(3.1) $$W_\mu\left\{f\left(\frac{1}{t}\right); p\right\}=h(p; \mu)$$

then

(3.2) $$L\left\{t^{2\mu-2} h\left(\frac{1}{t^2; \mu}\right); p\right\}=2^\mu p^{-\mu} K_{\mu-\frac{1}{2}}\{t^{-\mu-2} f(t^2)\, p\},$$

provided that the Weyl integral of $\left|f\left(\frac{1}{t}\right)\right|$ *exist,* $\operatorname{Re}\mu>0$ *and* $\operatorname{Re}p>0$.

Proof: Since we have [2, p. 188 (23)],

(3.3) $$R_\mu\{t^{-\mu-\frac{1}{2}} e^{-at^{-\frac{1}{2}}}; p\}=2^{\mu+\frac{1}{2}} \pi^{-\frac{1}{2}} a^{\frac{1}{2}-\mu} p^{\frac{1}{2}\mu-\frac{3}{4}} K_{\mu-\frac{1}{2}}(ap^{-\frac{1}{2}}).$$

Using the relations (3.1) and (3.3) in (1.6) we obtain (3.2) after little simplification.

Example: If we take

$$f\left(\frac{1}{t}\right)=t^{1-\alpha}\Phi_3\left(\alpha-1, \gamma; \frac{x}{t}, y\right)$$

then [5]

(3.4) $$W_\mu\left\{f\left(\frac{1}{t}\right); p\right\} = \frac{\Gamma(\alpha-\mu-1)}{\Gamma(\alpha-1)} p^{1+\mu-\alpha} \Phi_3\left(\alpha-\mu-1, \gamma; \frac{x}{p}, y\right)$$
$$= h(p; \mu)$$

for $\operatorname{Re}(\alpha-\mu-1)>0$, $\operatorname{Re}(\mu)>0$ and $R(p)>0$.

Putting these values of $f(t^2)$ and $h(t^{-2}; \mu)$ in relation (3.2) and using [1, p. 223] we obtain

(3.5) $$K_{\mu-\frac{1}{2}}\{t^{2\alpha-\mu-4} \Phi_3(\alpha-1, \gamma; xt^2, y); p\}$$
$$= \frac{2^{-\mu} p^{\mu-2\alpha+3} \Gamma(2\alpha-3) \Gamma(\alpha-\mu-1)}{\Gamma(\alpha-1)} \Xi_1\left(\alpha-\frac{3}{2}, \alpha-\mu-1, \alpha-1, \gamma; y; \frac{4x}{p^2}\right)$$

for $\operatorname{Re}(\alpha-\mu-1)>0$, $\operatorname{Re}(p)>0$, and $Re\,\mu>0$.

When $\mu=1$, (3.5) reduces to a known result [1, p. 223].

Theorem II. *If*

(3.6) $$W_\mu\{f(t); p\} = h(p; \mu)$$

then

(3.7) $$J_\nu\{t^{\nu+\frac{1}{2}} h(t^2; \mu); p\} = 2^\mu p^{-\mu} J_{\nu+\mu}\{t^{\mu+\nu+\frac{1}{2}} f(t^2); p\}$$

provided that the Weyl integral of $|f(t)|$ *exist* $\operatorname{Re}(\mu)>0$ *and* $\operatorname{Re}\nu>-1$.

Proof: Since we have [1, p. 194 (63)]

(3.8) $$R_\mu\{t^{\frac{1}{2}} J_\nu(at^{\frac{1}{2}}); p\} = 2^\mu p^{\frac{1}{2}\mu+\frac{1}{2}\nu} a^{-\mu} J_{\nu+\mu}(ap^{\frac{1}{2}}),$$

where $Re\,\mu>0$ and $Re\,\nu>-1$.

Using (3.6) and (3.8) in relation (1.5) we obtain (3.7) after little simplification.

REFERENCE

[1] A. Erdélyi et al. *Tables of integral transforms* Vol. I, McGraw-Hill, New York (1954).

[2] A. Erdélyi, et al. *Tables of integral transforms*, Vol. II, McGraw-Hill, New York (1954).

[3] C. S. Meijer, *Über eine Erweiterung der Laplace Transformation*, Proc. Nedrl. Akad. Wet., Amsterdam, 43, 599—608 (1940).

[4] S. L. Kalla, *Some theorems of fractional integration*, Proc. of Nat. Acad of Sc. India Vol. 36 Part IV, 1007—1012 (1966).

[5] S. L. Kalla, *Some theorems of fractional integration-II*, Proc of Nat Acad of Sc. India (to appear).

Department of Maths,
University of Jodhpur
Jodhpur, India.
and
Department of Maths,
McGill University,
Montreal, Canada.