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THE SPECTRAL METHOD FOR DETERMINING THE NUMBER OF TREES

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We consider finite, undirected graphs without loops or multiple edges.

1. A tree is an acyclic connected graph. It is known that a tree with n vertices contains $n-1$ edges.

This paper deals with determining the number of trees which are, as spanning subgraphs, contained in a given graph. Some of the trees which are counted in this number are mutually isomorphic. The number of trees of the graph G we denote by $D(G)$. Note that $D(G)=0$ if G is an unconnected graph.

A classical result in this area of graph theory was given by A. Cayley [1], who proved that for a complete graph G with n vertices the formula

$$D(G)=n^{n-2} \tag{1}$$

holds. Later on several proofs of this formula appeared. The survey of their different proofs is given in [2]. A survey of results which are related to determination of the number of trees in incomplete graphs is given, for example, in [3], [5] and [6].

The number of trees is of interest in several applications. A. K Kel'mans, for example, lists in [3], as an illustration of this fact, applications in the following areas: analysis of electrical networks, determination of the probability of connectedness of a telecommunicating system and analysis of maser's effects.

2. For the determination of the number of trees of a graph there is a matrix technique.

Let G have vertices $x_1, \ldots, x_n$ whose degrees are $d_1, \ldots, d_n$ respectively. Let $A=\| a_{ij} \|_1^n$ be the adjacency matrix and $D=\| d_{ij} \|_1^n$ the matrix of vertex degrees of the graph G. In the usual manner, for A we have $a_{ij}=1$ if x_i and x_j are adjacent and $a_{ij}=0$ if x_i and x_j are not adjacent. Further, we have $d_{ij}=d_i \delta_{ij}$, where δ_{ij} denotes the Kronecker's δ-symbol.

It is known that the number of trees is equal to the minor of arbitrary diagonal element of the matrix $D-A$. This result have been obtained independently by G. Kirchhoff [7], H. M. Trent [8] and some other authors. One proof of this statement can be found in [9].

Using described technique, the problem can be solved, in general case, only in principle, since by effective solving one finds difficulties while developing the determinant by use of which the number of trees is determined.

There are two modifications of this matrix method which, as we shall see, have the importance of independent methods.

1° In [3] for the graph G with n vertices the following function is introduced:

$$B_\lambda^n(G) = \frac{1}{\lambda} \det(D - A + \lambda I). \tag{2}$$

Note that for every graph $D-A$ is a singular matrix and that $B_\lambda^n(G)$ is a polynomial. It can easily be seen that the formula

$$D(G) = \frac{1}{n} B_0^n(G) \tag{3}$$

holds.

2° It is noticed in [9] that the number $D(G)$ can be expressed, for regular graphs, by use of the characteristical polynomial $P_G(\lambda) = \det(\lambda I - A)$ of the graph G. We have

$$D(G) = \frac{1}{n} P_G'(r), \tag{4}$$

where r denotes the index (degree) of the regular graph G.

Formulas (3) and (4) are simple consequences of the mentioned theorem of Kirchhoff — Trent. The importance of these formulas lies in the fact, that the functions $B_\lambda^n(G)$ and $P_G(\lambda)$ can be determined for some graphs by the use of the same functions for some simpler graphs.

Possibilities given by (3) are shortly described in [3].

In **4**, using the formula (4), we have determined the number of trees in some regular graphs. Part **4.** represents the main part of this paper.

3. The direct sum $G_1 \dot{+} G_2$ of graphs G_1 and G_2 is the graph which contains, as components of connectivity, all the components of the graphs G_1 and G_2.

The complete product $G_1 \nabla G_2$ of graphs G_1 and G_2 is obtained from the graph $G_1 \dot{+} G_2$, if each of the vertices of G_1 is joined by one edge with each of the vertices of G_2.

This type of the sum and product was considered in [10]. According to this paper, a graph is called elementary if it is connected and if it is ∇-primitive (i. e. if it cannot be represented as ∇-product of two graphs).

The following formulas are deduced in [3]:

$$B_\lambda^n(\overline{G}) = (-1)^{n-1} B_{-(\lambda+n)}^n(G), \tag{5}$$

$$B_\lambda^{n_1+n_2}(G_1 \dot{+} G_2) = \lambda B_\lambda^{n_1}(G_1) B_\lambda^{n_2}(G_2), \tag{6}$$

$$B_\lambda^{n_1+n_2}(G_1 \nabla G_2) = (\lambda + n_1 + n_2) B_{\lambda+n_2}^{n_1}(G_1) B_{\lambda+n_1}^{n_2}(G_2), \tag{7}$$

where $\overline{G}$ denotes the complement of the graph G.

Using formulas (5) — (7) it is possible to determine the functions $B_\lambda^n(G)$ for all the graphs if these functions are known for the elementary graphs. Since the quantity $D(G)$ can be simply determined on the basis of $B_\lambda^n(G)$, the same statement holds also for $D(G)$.

Unfortunately, according to [11], p. 508, the class of elementary graphs is very large. Nevertheless, many graphs can be represented by the use of the operations $\dot{+}$ and ∇, starting from a very narrow class of elementary graphs.

In [4] the function $B_\lambda^n(G)$ is determined if elementary graph G represents: a) graph containing only one isolated vertex, b) cycle of length m, c) path of length s.

On the basis of these three results only it is possible to determine by the use of formulas (5) — (7) the functions $B_\lambda^n(G)$, and in this way also the number of trees, for several kinds of graphs which are partially treated by a number of authors. So it is, for example, a routine job to determine $D(G)$ if the complement $\overline{G}$ of G contains, as the components, isolated vertices, complete graphs, bicomplete graphs, paths and cycles.

In that way, the method with the function $B_\lambda^n(G)$ gives all the results, related to the number of trees in several graphs, which are described in papers [5] and [6], i.e. almost all the results which are known in this area.

4. In the same way as the function $B_\lambda^n G)$ can be expressed using the same function of elementary graphs, so the function $P_G(\lambda)$ for some graphs G can be determined by characteristical polynomials of some simpler graphs. In this case the terms „simple“ and „compound“ graph are taken in relation to some operations on graphs which are not identical with those described in **3.**

We shall consider n-ary operations on graphs which have been introduced in [12] and [13]. These are the incomplete extended p-sum (shortly: NEPS, according to Serbocroat: nepotpuna proširena p-suma) of graphs and the Boolean function of graphs. We give for the NEPS the definition which is equivalent to one from [12] and which is mentioned in [14].

Let B be a set of n-tuples $(\beta_1, \ldots, \beta_n)$ of symbols 0 and 1 not containing n-tuple $(0, \ldots, 0)$.

Definition 1. NEPS $g(G_1, \ldots, G_n)$ of the graphs $G_1, \ldots, G_n$ with the basis B is the graph, whose set of vertices is equal to the Cartesian product of the sets of vertices of the graphs $G_1, \ldots, G_n$ and in which two vertices $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are adjacent if and only if there is a n-tuple $(\beta_1, \ldots, \beta_n)$ in B, such that $x_i = y_i$ holds exactly when $\beta_i = 0$ and x_i is adjacent to y_i in G_i exactly when $\beta_i = 1$.

Definition 2. Let $G_i = (X_i, U_i)$ $(i = 1, \ldots, n)$ be given graphs, where X_i and U_i denote corresponding sets of vertices and of edges. If $f(p_1, \ldots, p_n)$ is an arbitrary Boolean function $(f: \{0,1\}^n \to \{0,1\})$, the Boolean function $G = f(G_1, \ldots, G_n)$ of the graphs $G_1, \ldots, G_n$ is the graph $G = (X, U)$ where $X = X_1 \times \cdots \times X_n$ and where U is defined in the following way. For arbitrary two vertices $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ from G the Boolean variables $p_1, \ldots, p_n$ are defined, where, for every i, $p_i = 1$ if and only if x_i is adjacent to y_i in G_i. The vertices $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are adjacent in G if and only if, for every i, $x_i \neq y_i$ and $f(p_1, \ldots, p_n) = 1$.

The set of n-tuples, for which the Boolean function $f(p, \ldots, p_n)$ takes the value 1, we denote by F. We utilise also the abbreviation $\beta = (\beta_1, \ldots, \beta_n)$.

The introduced operations are very general. Varying the sets B and F one obtains several n-ary operations. Some of those special cases represent the operations on graphs which are known in literature.

For the NEPS we have the following theorem [14]:

Theorem 1. *The NEPS with the basis B of the graphs $G_1, \ldots, G_n$, whose spectrums are determined by $\{\lambda_{ji_j} \mid i_j = 1, \ldots, m_j\}$ $(j=1, \ldots, n)$, has the spectrum $\{\Lambda_{i_1, \ldots, i_n} \mid i_j = 1, \ldots, m_j,\ j=1, \ldots, n\}$, where*

$$\Lambda_{i_1, \ldots, i_n} = \sum_{\beta \in B} \lambda_{1 i_1}{}^{\beta_1} \cdots \lambda_{n i_n}{}^{\beta_n}.$$

The corresponding theorem for the Boolean function of graphs is proved in [13]. If the spectrum of G_j contains the numbers λ_{ji_j} $(i_j = 1, \ldots, m_j)$, the numbers from the spectrum of $\overline{G}_j$ will be denoted by $\overline{\lambda}_{ji_j}$. If G_j is a regular graph and if the sequence $\lambda_{j1}, \ldots, \lambda_{jm_j}$ is monotone and nonincreasing, it can be taken that the relations: $\overline{\lambda}_{j1} = m_j - 1 - \lambda_{j1}$, $\overline{\lambda}_{ji_j} = -1 - \lambda_{ji_j}$ $(i_j > 1)$ hold, which is proved in [15]. We also use the convention $\lambda_{ji_j}{}^{[\beta_j]} = \lambda_{ji_j}$ for $\beta_j = 1$ and $\lambda_{ji_j}{}^{[\beta_j]} = \overline{\lambda}_{ji_j}$ for $\beta_j = 0$.

Theorem 2. *If $G_1, \ldots, G_n$ are regular graphs, the spectrum of the graph $G = f(G_1, \ldots, G_n)$ is the set $\{\Lambda_{i_1, \ldots, i_n} \mid i_j = 1, \ldots, m_j,\ j = 1, \ldots, n\}$, where*

$$\Lambda_{i_1, \ldots, i_n} = \sum_{\beta \in F} \lambda_{1 i_1}{}^{[\beta_1]} \cdots \lambda_{n i_n}{}^{[\beta_n]}.$$

Formula (4) holds, as it was mentioned, if G is a regular graph. Thus, for the application of the formula (4) the fact, whether or not the NEPS or the Boolean function or regular graphs is a regular graph, is essential. The answer is positive and this fact can be easily deduced on the basis of the definition of the mentioned operations. An algebraic proof for the Boolean function, which can be extended to the NEPS, is given in [14].

We shall determine the number of trees for some classes of graphs, which appear as results of application of the NEPS and of the Boolean function on regular graphs. Since Theorems 1 and 2 give spectrums of graphs, wa shall modify formula (4) in the following way:

Theorem 3. *If $\{\lambda_1 = r, \lambda_2, \ldots, \lambda_m\}$ is the spectrum of the regular graph G of degree r, the following formula holds*:

$$D(G) = \frac{1}{m} \prod_{i=2}^{m} (r - \lambda_i). \tag{8}$$

We describe in more detail some special cases of the introduced operations.

Consider first the sum and product of graphs. These binary operations are of the type of NEPS. For the sum we have $B = \{(0, 1), (1, 0)\}$ and for the product $B = \{(1, 1)\}$. Let $G_1 = (X_1, U_1)$, $G_2 = (X_2, U_2)$, $G_1 + G_2 = (X, U)$ and $G_1 \times G_2 = (X, V)$. According to the definition we have $X = X_1 \times X_2$. Let $(x_1, x_2) \in X$, $(y_1, y_2) \in X$. For the sum the vertices (x_1, x_2) and (y_1, y_2) are adjacent if and only if either $x_1 = y_1$ and $(x_2, y_2) \in U_2$ or $(x_1, y_1) \in U_1$ and $x_2 = y_2$. For the product the mentioned vertices are adjacent if and only if $(x_1, y_1) \in U_1$ and $(x_2, y_2) \in U_2$.

The strong product of graphs is the NEPS with the basis containing all the n-tuples except n-tuple $(0, \ldots, 0)$.

Now we determine the number of trees in several regular graphs:

4.1. *Graph of the k-dimesional lattice.* The graph G of the k-dimensional lattice has, as the vertices, all the points with integer-coordinates from a cube of the k-dimensional Euclidean space, where two vertices are adjacent if and only if the corresponding points differ in exactly one coordinate. This graph can be represented as the sum of k complete graphs with n vertices. It is known that the spectrum of a complete graph with n vertices contains the number $n-1$ as well as $n-1$ numbers equal to -1 (see, for example, [15]). According to Theorem 1, the spectrum of the graph $G_1+\cdots+G_k$ contains all the numbers of the form $\lambda_{1i_1}+\cdots+\lambda_{ki_k}$. Using mathematical induction on k, it can easily be proved that the spectrum of the graph G contains the numbers $\lambda_i=n(k-i)-k$, $i=0, 1, \ldots, k$, with multiplicities $p_i=\binom{k}{i}(n-1)^i$. According to (8) we have

$$D(G)=n^{n^k-k-1}\prod_{i=1}^{k} i^{\binom{k}{i}(n-1)^i}.$$

4.2. *Graph of the prism.* Graph G of the prism is the graph whose vertices and edges correspond to the vertices and edges of the prism. Let the basis of the prism be a n-gon. Graph G can be represented as the sum of a cycle of length n and a complete graph K_2 with two vertices. The cycle of length n contains in the spectrum numbers $2\cos\frac{2\pi}{n}i$ $(i=0, 1, \ldots, n-1)$, and the spectrum of K_2 is equal to $\{-1, 1\}$. Thus, in the spectrum of G we find the numbers $2\cos\frac{2\pi}{n}i+1$, $2\cos\frac{2\pi}{n}i-1$, $i=0, 1, \ldots, n-1$, and we have

$$D(G)=\frac{1}{2n}\left[\prod_{i=1}^{n-1}\left(2-2\cos\frac{2\pi}{n}i\right)\right]\prod_{i=0}^{n-1}\left(4-2\cos\frac{2\pi}{n}i\right)$$
$$=\frac{8^{n-1}}{n}\prod_{i=1}^{n-1}\sin^2\frac{\pi i}{n}\left(1+2\sin^2\frac{\pi i}{n}\right).$$

4.3. *Square lattice on the torus.* Consider a circle torus. The circle, which lies on the torus and whose plane is normal on the axis of the torus, is called the horizontal circle. Vertical circle is obtained when the torus is cut by a semi-plane which starts from the axis of the torus. If we have some horizontal and some vertical circles on the torus, we obtain a square lattice. The graph G of the square lattice has, as vertices, intersections of horizontal and vertical circles. Adjacent are those vertices which are immediately joined by the arc of one of the circles which lie on the torus.

The graph G can be represented as the sum of two cycles. Let those be cycles of length m and n (m horizontal and n vertical circles). According to the previous statements one obtains

$$D(G)=\frac{1}{mn}\prod_{i=0}^{n-1}\prod_{j=0}^{m-1}\left(4-2\cos\frac{2\pi}{n}i-2\cos\frac{2\pi}{m}j\right)$$
$$=\frac{4^{mn-1}}{mn}\prod_{i=0}^{n-1}\prod_{j=0}^{m-1}\left(\sin^2\frac{\pi}{n}i+\sin^2\frac{\pi}{m}j\right),\quad (i, j)\neq(0, 0).$$

If we consider, instead of the sum, the strong product of the cycles, we obtain the graph which corresponds to the square lattice on the torus in which every „square“ also has the „diagonals“ constructed. On the basis of

Theorem 1, the strong product of the graphs G_1 and G_2 contains in the spectrum all the numbers of the form $\lambda_{1i_1}+\lambda_{2i_2}+\lambda_{1i_1}\lambda_{2i_2}$. Thus, for the described modification of the square lattice on the torus we have

$$D(G)=\frac{1}{mn}\prod_{i=0}^{n-1}\prod_{j=0}^{m-1}\left(8-2\cos\frac{2\pi}{n}i-2\cos\frac{2\pi}{m}j-4\cos\frac{2\pi}{n}i\cos\frac{2\pi}{m}j\right)$$

$$=\frac{2^{mn-1}}{mn}\prod_{i=0}^{n-1}\prod_{j=0}^{m-1}\left(4-\cos\frac{2\pi}{n}i-\cos\frac{2\pi}{m}j-2\cos\frac{2\pi}{n}i\cos\frac{2\pi}{m}j\right),\quad (i,j)\neq(0,0).$$

4.4. *A graph of the cyclic structure.* Let the graph G has the following structure. The set of the vertices of G is partitioned in $m\,(m\geqslant 3)$ subsets $S_1,\ldots,S_m$, each containing n vertices. For every $i=2,\ldots,m-1$ every vertex from S_i is adjacent to every vertex from S_{i-1} and to every vertex from S_{i+1}. Besides this, adjacent is every vertex from S_1 to every vertex from S_m. Other pairs of vertices are not adjacent.

G is representable as the product of a cycle with m vertices and of a complete graph with n vertices and with one loop added to each vertex. The first graph has the spectrum $\left\{2\cos\frac{2\pi}{m}i \,\middle|\, i=0,1,\ldots,m-1\right\}$; for the second, the spectrum contains the number n as well as $n-1$ numbers equal to 0.

G has mn vertices and is of degree $2n$, so that we have

$$D(G)=\frac{1}{mn}\left[\prod_{i=1}^{m-1}\left(2n-2n\cos\frac{2\pi}{m}i\right)\right]\prod_{i=0}^{m-1}(2n-0)^{n-1}$$

$$=\frac{2^{mn+m-2}\,n^{mn-2}}{m}\prod_{i=1}^{m-1}\sin^2\frac{\pi}{m}i.$$

4.5. *The graph of the diagonals in space of a prism.* Finally consider the graph G which has, as vertices, vertices of a n-sided prism and as edges all the diagonals in space of the prism. G can be represented as exclusive disjunction of the cycle G_1 of length n and of the complete graph G_2 with two vertices.

For the exlusive disjunction we have $F=\{(1,0),(0,1)\}$ and its spectrum contains, according to Theorem 2, all the numbers of the form $\lambda_{1i_1}\overline{\lambda}_{2i_2}+\overline{\lambda}_{1i_1}\lambda_{2i_2}$.

The spectrum of G_2 is $\{-1,1\}$ and for $\overline{G}_2$ we have $\{0,0\}$. Thus, the spectrum of G contains $2n$ numbers $\overline{\lambda}_{11},\ldots,\overline{\lambda}_{1n},-\overline{\lambda}_{11},\ldots,-\overline{\lambda}_{1n}$. According to what has been said before, the complement of a cycle contains in the spectrum the numbers $n-3$ and $-1-2\cos\frac{2\pi}{n}i$ $(i=1,\ldots,n-1)$. So we have

$$D(G)=\frac{n-3}{n}\prod_{i=1}^{n-1}\left(n-4-2\cos\frac{2\pi}{n}i\right)\left(n-2+2\cos\frac{2\pi}{n}i\right).$$

5. The characteristical polynomial, and the spectrum, of a graph can be defined respectively as the characteristical polynomial, and the spectrum, of some square matrix which is in a certain way associated to the graph. The spectral method in the theory of graphs is the set of procedures for obtaining and proving statements about the graph structure, which use, in their essential points, the characteristical polynomials, or the spectra of graphs.

In **3.** and **4.** two methods for determining the number of trees in graphs were described. Both methods are spectral. As it has been said, by the use of these methods almost all the known and some other results in considered area can be obtained. The spectral method, therefore, can be considered as a very efficient one in the considered field of the graph theory.

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