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TOPOLOGIZING THE HYPERSETS[1)]

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Introduction

In this paper we investigate some topologies on closed subsets of a topological space. In introducing a topology, we use a family of subsets that we call a topologizing system (and have it denoted by $\mathcal{A}$), resuming so a point of view that, in dealing with hyperspaces, a topologizing system is what matters, not the family of closed subsets. So we try to develop further an idea initiated in [7].

In section **1.**, we consider several mappings between the families of sets and state some basic facts without proofs (which, for instance, can be found in [6] and [8]). We only prove 1.5 and 1.6.

In section **2.**, we follow the construction of the space $\varkappa(X)$ (this notation is after [10]), being introduced first by A. Tychonoff [10] and studied by V. I. Ponomarov in [10]. Varying a topologizing system, the spaces $\varkappa(X, \mathcal{A})$ have all essential properties of $\varkappa(X)$ as they were established in [10] or, in the case of such a minimal system, in [3] (simple Funktionentopologie). That this generality is not only formal can be seen from 2.10, 2.11 and 2.13.

In section **3**, we establish some of the properties of the spaces $\lambda(X, \mathcal{A})$ which, in the case when the topologizing system is the family of all open sets of X, are reduced to $\lambda(X)$ (this notation is again after [10]). Giving first a λ-characterization of the families of finite character, a result (3.6) is proved which formaly extends the Tukey lemma. A realization of the topological product as a subspace of a $\lambda(X, \mathcal{A})$, with the "small" open sets in $\mathcal{A}$, is also given.

In the last section **4**, we establish a set theoretic formula and prove a proposition in connection with it (4.5), which is logically equivalent to the Alexander lemma. For X compact $\lambda(X)$ is compact what is implied by this proposition and, in a set theoretical form, that is what the proposition actually means.

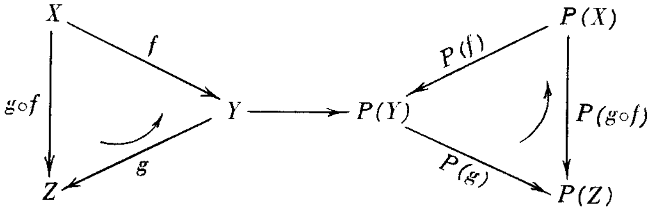
1. Some mappings of families of sets.

Let $\mathcal{S}$ be the category of all sets and all mappings. Define the *P-functor* P: $\mathcal{S} \to \mathcal{S}$ in the following way: For an object $X \in \mathcal{S}$, $P(X)$ is the partitive set of X and for a morphism $f: X \to Y$ in $\mathcal{S}$ let $P(f): P(X) \to P(Y)$ be such that for $A \in P(X)$,

$$P(f)(A) = \{y \in Y: y = f(x), \text{ for some } x \in A\}.$$

1) I wish to express my gratitude to professor Dragiša Ivanović. Without his kind help the work on this paper would have been impossible.

Then, P is a covariant functor, what can be represented by the following diagram



and $P(\mathcal{S})$ is a subcategory of $\mathcal{S}$ (though not a full subcategory).

For $X \in \mathcal{S}$, let $P^2(X) = P(P(X))$ and $P^2(X)$ is called the second partitive set of X. The elements of $P(X)$ will be denoted by $A, B, \ldots F, \ldots$ and called subsets of X and those of $P^2(X)$ by $A^{(1)}, B^{(1)}, \ldots F^{(1)}, \ldots$, called families of subsets of X.

A *union mapping* $u: P^2(X) \to P(X)$ is defined so that

$$u(A^{(1)}) = \{x : x \in A, \text{ for some } A \in A^{(1)}\}.$$

We will also use the symbol $|A^{(1)}|$ to denote $u(A^{(1)})$. The set $u(A^{(1)})$ is called the body of the family $A^{(1)}$. Let $i: X \to P(X)$ be the mapping $i(x) = \{x\}$.

The set $P(X)$ is partially ordered by inclusion and we use the same symbol $\subseteq$ to denote the order relation in $P(X)$ and $P^2(X)$ what is justified by

1.1. *If* $A^{(1)} \subseteq B^{(1)}$, *then* $u(A^{(1)}) \subseteq (B^{(1)})$.

The object (and subobjects) of $P(\mathcal{S})$ and $P^2(\mathcal{S})$ have "more structure" what allows one to define some natural mappings and relations which are not definable in $\mathcal{S}$.

The *complement mapping* $C: P(X) \to P(X)$ is defined by $C(A) = X \setminus A$, and it is evident that $C \circ C = \text{id}$. Two mappings $C: P^2(X) \to P^2(X)$ and $P(C): P^2(X) \to P^2(X)$ are distinct and should not be confused. If $A \subseteq X$, write $\langle A \rangle$ to denote $P(A)$. So

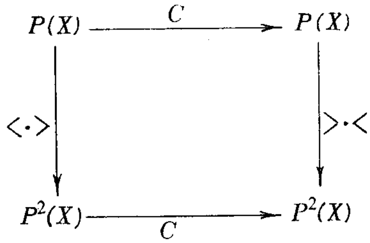
$$\langle A \rangle = \{F \in P(X) : F \subseteq A\}.$$

is a mapping of $P(X)$ into $P^2(X)$. The following properties of the mapping $\langle \cdot \rangle$ are easily established.

1.2. (I) $A \subseteq B \Rightarrow \langle A \rangle \subseteq \langle B \rangle$, (II) $\langle A \cap B \rangle = \langle A \rangle \cap \langle B \rangle$,

(III) $A \neq B \Rightarrow \langle A \rangle \neq \langle B \rangle$.

Since the mapping C is an epimorphism there is a unique mapping $\rangle \cdot \langle$ such that the following diagram commutes



Now we have

1.3. (I) $\langle\cdot\rangle = C\circ\rangle\cdot\langle\circ C$, (II) $\rangle\cdot\langle = C\circ\langle\cdot\rangle\circ C$

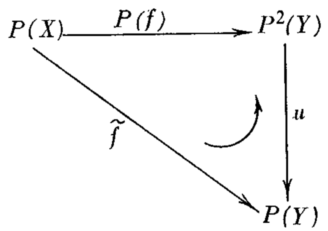
(III) $\rangle A\langle = \{F\in P(X): F\cap A\neq\emptyset\}$, (IV) $\rangle A\langle \sqsupseteq \langle A\rangle$.

Using 1.2 and applying 1.3 (II), we also have

1.4. (I) $A\subseteq B \Rightarrow \rangle A\langle \subseteq \rangle B\langle$, (II) $\rangle A\cup B\langle = \rangle A\langle \cup \rangle B\langle$,

(III) $A\neq B \Rightarrow \rangle A\langle \neq \rangle B\langle$.

To denote the set of all singletons $\{\{x\}: x\in X\}$, we will use X and for the set X taken as an element of $P(X)$, the symbol $\{X\}$. Consider two mappings $\Phi_i: P(X)\to P(Y)$, $i=1, 2$. We will write $\Phi_1\subseteq\Phi_2$ if $\Phi_1(A)\subseteq\Phi_2(A)$ for each $A\in P(X)$. For a mapping $\Phi: P(X)\to P(Y)$, let $f=\Phi\,|\,X: X\to P(X)$ and let $\tilde f$ be the mapping defined by the diagram



The mapping Φ is *monotone* if $A\subseteq B$ in $P(X)$ implies $\Phi(A)\subseteq\Phi(B)$ in $P(Y)$. Now we can prove.

1.5. *If the mapping* $\Phi: P(X)\to P(Y)$ *is monotone then* $\tilde f\subseteq\Phi$.

Proof. Let $A\in P(X)$ and $Y\in\tilde f(A)=uP(f)(A)$. Then $y\in P(f)(x)=$ $=\Phi(x)$, for some $x\in A$. Since $\{x\}\subseteq A$ and Φ is monotone, we have $\Phi(x)\subseteq$ $\subseteq\Phi(A)$ and so $y\in\Phi(A)$. Thus $\tilde f(A)\subseteq\Phi(A)$.

1.6. *Let* $X\subseteq A^{(1)}\subseteq P(X)$ *and* $Y\subseteq B^{(1)}\subseteq P(Y)$ *and let*

$$\Phi: A^{(1)}\to B^{(1)}$$

be $1-1$ *and onto and let both* Φ *and* Φ^{-1} *be monotone. Then* $\tilde f: X\to Y$ *is* $1-1$ *and onto and* $\tilde f=\Phi$.

Proof. Since $Y\subseteq B^{(1)}$, for $y\in Y$ there is an $x\in X$ such that $y\in\Phi(x)$. From $\{y\}\subseteq\Phi(X)$, it follows that $\Phi^{-1}(y)\subseteq\{x\}$ what implies $\Phi^{-1}(y)=x$. Being Φ $1-1$ we have $\Phi(x)=y$. Hence $f=\Phi\,|\,X: X\to Y$ is $1-1$ and onto. Since Φ^{-1} is monotone, according to 1.5, for $B\in B^{(1)}$ we have

$$\tilde f^{-1}(B)\subseteq\Phi^{-1}(B)=A$$

or being $\tilde f$ monotone,

$$\tilde f(\tilde f^{-1}(B))=B=\Phi(A)\subseteq\tilde f(A).$$

So $\Phi\subseteq\tilde f$ and $\tilde f\subseteq\Phi$ imply $\Phi=\tilde f$.

2. The space $\varkappa(X, \mathcal{A})$.

Let X be a topological T_1-space and $\mathcal{A}$ a family of subsets of X containing the family $\bar{\mathcal{A}}$ having for its members all complements of the singletons $\{X \setminus \{x\} : x \in X\}$ as well as X itself. The family $\mathcal{A}$ will be called the *topologizing system*. To the pair $(X, \mathcal{A})$ we correspond the topological space $\varkappa(X, \mathcal{A})$ having for its elements all closed subsets of X (the empty set $\emptyset$ included), and for the open subbase of its topology the collection of all

$$\langle U \rangle : U \in \mathcal{A}.$$

In the case $\mathcal{A}$ equals the topology $\mathcal{T}_x$ of X, we write $\varkappa(X)$ instead of $\varkappa(X, \mathcal{A})$.

2.1. *Let $\tilde{\mathcal{A}}$ be the family of all finite inersections of the members $\mathcal{A}$. Then*

$$\varkappa(X, \tilde{\mathcal{A}}) = \varkappa(X, \mathcal{A})$$

Proof. Follows from 1.2 (II).

So we can suppose that $\langle U \rangle : U \in \mathcal{A}$ is a base for $\varkappa(X, \mathcal{A})$ replacing $\mathcal{A}$ by $\tilde{A}$ if necessary.

2.2. *There exist topologically equivalent bases $\mathcal{A}_1$ and $\mathcal{A}_2$ for some topology on X and yet*

$$\varkappa(X, \mathcal{A}_1) \not\approx \varkappa(X, \mathcal{A}_2).$$

Proof. Take X to be the set N of all natural numbers. Let

$$\mathcal{A}_1 = \{\{x\} : x \in N\} \cup \{N \setminus \{x\} : x \in N\} \cup \{N\}, \quad \mathcal{A}_2 = P(N).$$

Then the set $\langle 1, 2 \rangle$ is not open in $\varkappa(X, \mathcal{A}_1)$.

Let $i : X \to \varkappa(X, \mathcal{A})$ be such that $i(x) = \{x\}$. Then for $U \subseteq X : i(U) \subseteq \langle U \rangle$ and $i^{-1}(\langle U \rangle) = U$. So if $\mathcal{A} \subseteq \mathcal{T}_x$, the mapping i is continuous.

2.3. *The space $\varkappa(X, \mathcal{A})$ is a T_0-space.*

Proof. Let $F_1 \neq F_2$ and, say, $x \in F_1 \setminus F_2$. Then $\langle X \setminus \{x\} \rangle$ is a neighborhood of F_2 which does not contain F_1.

2.4. $F_1 \in \overline{\{F_2\}}$ *in* $\varkappa(X, \mathcal{A}) \Leftrightarrow F_1 \supseteq F_2$.

Proof. $\Rightarrow$: Let $F_1 \in \overline{\{F_2\}}$. If $F_1 = X$, then $F_1 \supseteq F_2$. If $F_1 \neq X$, take any $x \in X \setminus F_1$. Then $\langle X \setminus \{x\} \rangle$ is a neighborhood of F_1 and $F_2 \in \langle X \setminus \{x\} \rangle$. The last relation means that $F_2 \subseteq X \setminus \{x\}$. Therefore

$$F_2 \subseteq \cap \{X \setminus \{x\} : x \in X \setminus F_1\} = F_1.$$

$\Leftarrow$: Suppose $F_1 \supseteq F_2$ and let $\langle U \rangle$, $U \in \tilde{\mathcal{A}}$ be a basic open set containing F_1. Then $F_1 \subseteq U$ and so $F_2 \subseteq U$, or $F_2 \in \langle U \rangle$ what implies $F_1 \in \overline{\{F_2\}}$.

Notice that the empty set $\emptyset$ is everywhere dense in $\varkappa(X, \mathcal{A})$ and that $\emptyset$ need not be open if $\tilde{\mathcal{A}}$ does not contain $\emptyset$.

2.5. *If $f : \varkappa(X, \mathcal{A}) \to \varkappa(Y, \mathcal{B})$ is continuous, then f is monotone.*

Proof. By 2.4, if $F_1 \subseteq F_2$ then $F_2 \in \overline{\{F_1\}}$ and since f is continuous $f(F_2) \in f(\overline{\{F_1\}}) \subseteq \overline{\{f(F_1)\}}$. Using again 2.4, we have $f(F_1) \subseteq f(F_2)$.

2.6. *If* $f: \varkappa(X, \mathcal{A}) \to \varkappa(Y, \mathcal{B})$ *is a homeomorphism, then* f *defines a mapping* $g: X \to Y$ *which also is a homeomorphism.*

Proof. Immediately follows from 1.6. (The converse of 2.6 is, of course, false).

2.7. *The space* $\varkappa(X, \mathcal{A})$ *is compact and connected for each* X.

Proof. Let $F^{(1)}$ and $F^{(2)}$ be two non-empty, closed sets in $\varkappa(X, \mathcal{A})$. Then, according to 2.4, $\{X\} \in F^{(1)} \cap F^{(2)}$.

2.8. *If* $f: \varkappa(X, \mathcal{A}) \to \varkappa(X, \mathcal{A})$ *is continuous and onto, then* $f(\{X\}) = \{X\}$.

Proof. Let F be an arbitrary element of $\varkappa(X, \mathcal{A})$. Then $F \subseteq X$ and, according to 2.5, $f(F) \subseteq f(\{X\})$. Since f is onto, $f(F)$ is an arbitrary element of $\varkappa(X, \mathcal{A})$ what implies $f(\{X\}) = \{X\}$.

2.9. *If* X *is compact, then each continuous mapping* $f: \varkappa(X, \mathcal{A}) \to \varkappa(X, \mathcal{A})$ *has a fixed point.*

Proof. Let $\mathcal{F}$ be the family of those $F \in \varkappa(X, \mathcal{A})$ such that $f(F) \subseteq F$. $X \in \mathcal{F}$ and $\mathcal{F} \neq \emptyset$. Let $\{F: F \in \mathcal{L}\}$ be a chain in $\mathcal{F}$. Then, since X is compact

$$\emptyset \neq F_0 = \bigcap \{F: F \in \mathcal{L}\} \subseteq F.$$

So, by 2.5, $f(F_0) \subseteq f(F)$ for $F \in \mathcal{F}$. Further

$$f(F_0) \subseteq \bigcap \{f(F): F \in \mathcal{L}\} \subseteq \bigcap \{F: F \in \mathcal{L}\} = F_0,$$

and $F_0 \in \mathcal{F}$. According to the minimal principle, ([4], p. 33), there is a minimal element $\underline{F}$ of $\mathcal{F}$. Then $f(\underline{F}) \subseteq \underline{F}$, and since $\underline{F}$ is minimal $f(\underline{F}) = \underline{F}$.

Now let X be a set of cardinality τ taken with the discrete topology and let $\mathcal{A} = \underline{\mathcal{A}}$ ($\underline{\mathcal{A}}$ consists of X and of all complements of singletons). Denote, in this case, $\varkappa(X, \underline{\mathcal{A}})$ by $\varkappa(\tau)$ or $\varkappa(X)$.

On the other hand, let $F = \{0, 1\}$ be the space having for its open sets $\emptyset$, $\{0\}$ and F. The topological product of τ copies of F is denoted by F^τ. The universality of F for the category of T_0-spaces having the weight $\leqslant \tau$ was shown by P. S. Alexandrov in [1].

2.10. *The spaces* $\varkappa(\tau)$ *and* F^τ *are homeomorphic.*

Proof. The mapping $f: \varkappa(\tau) \to F^\tau$ given by

$$f(A) = \begin{cases} 1, & x \in A \\ 0, & x \notin A, \end{cases}$$

for each $A \subseteq X$ is obviously a homeomorphism. Another universal T_0-space in which any T_0-space of the weight $\leqslant \tau$ can be imbedded, is $\varkappa(D^\tau)$, where D^τ is the topological product of τ copies of the discrete space $D = \{0, 1\}$ (see [10]). A. N. Kolmogorov asked whether $\varkappa(D^\tau)$ and F^τ were homeomorphic, and that they are not was shown in [9] and [2].

2.11. *The spaces* $\varkappa(D^\tau)$ *and* F^τ *are not homeomorphic.*

Proof. By 2.6, the homeomorphism of $\varkappa(D^\tau)$ and $F^\tau \approx \varkappa(\tau)$ would imply the homeomorphism of D^τ and X, but they are not even of the same cardinality (card $X = \tau$).

2.12. *Let Y be a subspace of X. Then $\underline{\varkappa}(Y) \approx \langle Y \rangle$ taken as a subspace of $\underline{\varkappa}(X)$.*

Proof. If K is any finite subset of X, then basic open sets in $\underline{\varkappa}(Y)$ and $\langle Y \rangle$ are $\langle Y \setminus K \rangle$ and $\langle Y \rangle \cap \langle X \setminus K \rangle$ respectively. It is easy to see that $\langle Y \setminus K \rangle = \langle Y \rangle \cap \langle X \setminus K \rangle$.

The following result is a theorem due to P. S. Alexandrov ([1]) We will give a proof close to that of Alexandrov but carried out in the ambient space $\underline{\varkappa}(X)$.

2.13. *Each T_0-space of the weight $\leqslant \tau$ is homeomorphic to some subspace of $\varkappa(\tau)$.*

Proof. Let X be of the weight $\sigma \leqslant \tau$ and let Σ be a base of closed sets in X of cardinality σ. The mapping $f: X \to \underline{\varkappa}(\Sigma)$ given by

$$f(x) = \{F \in \Sigma : x \in F\}$$

is 1—1 because X is T_0-space and so there is an F containing one and not containing the other of each two points in X. Let $V = \langle \Sigma \setminus \{F_1, \ldots, F_n\} \rangle$ be a basic open set in $\underline{\varkappa}(\Sigma)$. Then

$$f^{-1}(V) = X \setminus \bigcup \{F_i : i = 1, \ldots, n\}$$

since for $x \in F_i$, $f(x)$ contains F_i. So f is continous. On the other hand $f: X \to f(X)$ is open for

$$f(U) = \langle \Sigma \setminus (X \setminus U) \rangle \cap f(X)$$

is open in $f(X)$ when U is open in X.

3. The space $\lambda(X, \mathcal{A})$.

Let X be a topological space and $\mathcal{A}$ a family of subsets of X containing at least the set X. The family $\mathcal{A}$ will be called again a *topogologizing system* Now to the pair $(X, \mathcal{A})$ we correspond the topological space $\lambda(X, \mathcal{A})$ having for its elements all closed subsets of X (the empty set excluded or if included is taken to be close and open) and for the open subbase of its topology the collection of all

$$\rangle U \langle : U \in \mathcal{A}.$$

In the case $\mathcal{A}$ equals the topology $\mathcal{T}_x$ of X, we write $\lambda(X)$ intead of $\lambda(X, \mathcal{A})$.

Since the mapping $\rangle \cdot \langle : P(X) \to P^2(X)$ has a natural extension to the set Σ of all finite families of members of $P(X)$, putting for $\sigma = \{U_1, \ldots, U_n\} \in \Sigma$ that

$$\rangle \sigma \langle = \rangle U_1, \ldots, U_n \langle = \rangle U_1 \langle \cap \cdots \cap \rangle U_n \langle,$$

then it follows that a standard base for $\lambda(X, \mathcal{A})$ will be

$$\rangle\sigma\langle : \sigma \in \Sigma(\mathcal{A}),$$

where $\Sigma(\mathcal{A})$ stands for the set of all finite families of members of $\mathcal{A}$.

3.1. *If* $F_1 \subseteq F_2$, *then* $F_1 \in \overline{\{F_2\}}$, *If* $\mathcal{A} \supseteq \mathcal{G}_x$ *then* $F_1 \in \overline{\{F_2\}}$ *implies* $F_1 \subseteq F_2$.

Proof. Suppose $F_1 \subseteq F_2$ and let $\rangle U_1, \ldots, U_n\langle$ be a basic open set containing F_1. Then $F_1 \cap U_i \neq \emptyset$, $i=1, \ldots, n$, and it follows that $F_2 \cap U_i \neq \emptyset$, so that $F_2 \in \rangle U_1, \ldots, U_n\langle$. Hence $F_1 \in \overline{\{F_2\}}$.

To prove the second part, suppose $F_1 \in \overline{\{F_2\}}$. Then $F_2 \supseteq F_1$, for otherwise $X \setminus F_2 \in \mathcal{A}$ and $F_1 \in \rangle X \setminus F_2\langle$, $F_2 \notin \rangle X \setminus F_2\langle$, what would contradict $F_1 \in \overline{\{F_2\}}$.

From 3.1 it follows that $\{X\}$ is everywhere dense in $\lambda(X, \mathcal{A})$.

3.2. *If* $f : \lambda(X) \to \lambda(Y)$ *is continuous, then* f *is monotone.*

Proof. By the first part of 3.1, $F_1 \subseteq F_2$ implies $F_1 \in \overline{\{F_2\}}$ and we have $f(F_1) \subseteq f(\overline{\{F_2\}}) \subseteq \overline{\{f(F_2)\}}$. Now, by the second part of 3.1, $f(F_1) \subseteq f(F_2)$.

3.3. *Let* X *and* Y *be* T_0*-spaces. If* $\lambda(X) \approx \lambda(Y)$ *then* $X \approx Y$.

Proof. Follows from 1.6 (The converse is, of course, true).

3.4. *The space* $\lambda(X, \mathcal{A})$ *is connected.*

Proof. Let $F_0^{(1)}$ and $F_1^{(1)}$ be two open disjoint sets of $\lambda(X, \mathcal{A})$ such that $\lambda(X, \mathcal{A}) = F_0^{(1)} \cap F_1^1$. If $\{X\} \in F_i^{(1)}$, then $\overline{\{X\}} = \lambda(X, \mathcal{A}) \subseteq F_i^{(1)}$ and so $F_{1-i}^{(1)}$ must be empty.

The space $\lambda(X)$ will have some other properties analoguous to $\varkappa(X, A)$ and being implied by 3.2.

In proceeding further, we give a „λ-topology" characterization of a family of finite character (see [4]). Before stating that characterization, recall that a family $\mathcal{A}$ of sets is of *finite character* if and only if each finite subset of a member of $\mathcal{A}$ is a member of $\mathcal{A}$, and each set A, every finite subset of which belongs to $\mathcal{A}$, itself belongs to $\mathcal{A}$.

3.5. *A family* $\mathcal{A} \subseteq P(X)$ *is of finite character* $\Leftrightarrow$ $\mathcal{A}$ *is a closed subset of* $\lambda(X)$, *where* X *is the discrete space.*

Proof. $\Rightarrow$: Suppose $\mathcal{A}$ is of finite character and let $F \notin \mathcal{A}$, where $F \in \lambda(X)$ (Two sets $P(X) \setminus \emptyset$ and $\lambda(X)$ are identical). Then there is a finite set $K = \{x_1, \ldots, x_n\}$ such that $K \subseteq F$ and $K \notin \mathcal{A}$, for otherwise every finite subset of F belongs to $\mathcal{A}$ and, according to the definition of a family of finite character F would belong to $\mathcal{A}$. K is not a subset of any $A \in \mathcal{A}$ since every finite subset of a member of $\mathcal{A}$ belongs to $\mathcal{A}$. The singletons are open in X so the set $\rangle x_1\langle \cap \rangle x_2\langle \cap \cdots \cap \rangle x_n\langle$ is open in $\lambda(X)$, contains F and does not contain any $A \in \mathcal{A}$.

$\Leftarrow$: Suppose $\mathcal{A}$ is closed in $\lambda(X)$. For $A \in \mathcal{A}$. $\overline{\{A\}} \subseteq \mathcal{A}$ and, according to 3.1, each subset and so each finite subset of A belongs to $\mathcal{A}$. Let $F \in \lambda(X)$ has the property that each finite subset of F belongs to $\mathcal{A}$. Let $\rangle U_1, \ldots, U_n\langle$

be a basic open set containing F. Let $x_i \in F \cap U_i$, $i=1, \ldots, n$. Then the set $\{x_1, \ldots, x_n\} \in \mathcal{A}$ and $\{x_1, \ldots, x_n\} \in \rangle U_1, \ldots, U_n \langle$. This implies $F \in \overline{\mathcal{A}} = \mathcal{A}$. Hence $\mathcal{A}$ is of finite character.

3.6. *Let $\mathcal{F}$ be a closed subset of $\lambda(X, \mathcal{A})$, where $\mathcal{A} \subseteq \mathcal{G}_x$. Then $\mathcal{F}$ has a maximal member.*

Proof. Let $\mathcal{Z} = \{F\}$ be a chain in $\mathcal{F}$. Let

$$A_0 = \cup \{F : F \in \mathcal{Z}\}$$

Then

$$\overline{A_0} \supseteq F, \text{ for each } F \in \mathcal{Z}.$$

Let $\rangle U_1, \ldots, U_n \langle$ be a basic neighborhood of $\overline{A_0}$. Let $x_i \in \overline{A_0} \cap U_i$, $i=1, \ldots, n$. Since $U_i \in \mathcal{A} \subseteq \mathcal{G}_x$ is an open neighborhood of x_i in X, there is an $y_i \in A_0$ such that $y_i \in U_i$ for each $i=1, \ldots, n$. Each y_i is an element of some $F_i \in \mathcal{Z}$ and let $F \in \mathcal{Z}$ be such that $F \supseteq F_i$, $i=1, \ldots, n$. Now, $F \in \rangle U_1, \ldots, U_n \langle$ and so $\overline{A_0} \in \overline{\mathcal{A}} = \mathcal{A}$. Applying the maximal principle we conclude that $\mathcal{A}$ has a maximal member.

When X is a discrete space and the topologizing system $\mathcal{A} = \mathcal{G}_x$, 3.6 reduces to the Tukey lemma. On the other hand 3.6 is a consequence of the Hausdorff maximal principle which is logically equivalent to the Tukey lemma. So we have.

3.7. *The statement* 3.6 *is logically equivalent to the Tukey lemma* (*and so to the axiom of choice*).

Note that we have taken the Tukey lemma in the form as given in [4]. According to the same book [4], p. 36 there is another more general formulation.

3.8. *The topological product hyperspace.*

Consider an arbitrarily given indexed family

$$\mathcal{F} = \{X_\zeta : \zeta \in \mathcal{Z}\}$$

of spaces. Let $X = \cup \{\zeta \times X : \zeta \in \mathcal{Z}\}$. Then for $\zeta' \neq \zeta''$, two sets $\zeta' \times X_{\zeta'}$ and $\zeta'' \times X_{\zeta''}$ are disjoint. For $U \subseteq X_\zeta$, denote by U_ζ the set $\zeta \times U$ and topologize X_ζ so that U_ζ is open if and only if U is open X_ζ. By this convention it follows that the subset $\zeta \times X_\zeta$ of X will be denoted by X_ζ.

Take X to have the discrete topology and the topologizing system $\mathcal{A}$ to be the family of all open sets in X_ζ for every $\zeta \in \mathcal{Z}$. Then $\lambda(X, \mathcal{A})$ will be called the *topological product hyperspace of the family* $\mathcal{F}$. Consider now the subspace

$$\Pi = \Pi \{X_\zeta : \zeta \in \mathcal{Z}\}$$

of $\lambda(X, \mathcal{A})$ given by

$$\Pi = \{A \in X : A \cap X_\zeta \text{ is a singleton for each } \zeta \in \mathcal{Z}\}.$$

On the other hand, let $\times \{X_\zeta : \zeta \in \mathcal{Z}\}$ be the topological product of the spaces $\{X_\zeta : \zeta \in Z\}$. Consider the mapping

$$\Gamma : \times \{X_\zeta : \zeta \in \mathcal{Z}\} \to \Pi$$

given by $\Gamma(x)=\{(\zeta, x(\zeta)):\zeta\in\mathcal{Z}\}$. It is obvious that Γ is 1—1 and onto. From

$$\Gamma^{-1}(\rangle U_\zeta\langle\cap\Pi)=\{x:x(\zeta)\in U\}$$

$$\Gamma\{x:x(\zeta)\in U\}=\rangle U_\zeta\langle\cap\Pi.$$

where $U_\zeta=\zeta\times U$ and $U\subseteq X_\zeta$, it follows that Γ is a homeomorphism. Hence the subspace Π of $\lambda(X, \mathcal{A})$ is just another realization of the topological product space what motivates us to call $\lambda(X, \mathcal{A})$ the topological product hyperspace.

4. Compactness of $\lambda(X, A)$.

In what follows we will prove several set theoretic formulas connected with the formalism involved in the consideration of compactness of $\lambda(X, \mathcal{A})$. We will also consider the logical equivalence of some propositions and now we will cite some of the known equivalences that we will need here.

According to [5], the Tychonoff theorem (the topological product of compact spaces is compact) is equivalent to the axiom of choice (*AC*) (see also [12]).

According to [13], the following three propositions are logically equivalent:

Alexander lemma: If $\mathcal{S}$ is a subbase for the topology of a space X such that every cover of X by membres of $\mathcal{S}$ has a finite subcover, then X is compact.

Axiom of choice for families of finite sets (*ACF*): There is a choice function for every family of non-empty finite sets.

Tychonoff theorem for finite spaces (*TF*): The topological product of a family of finite spaces is compact.

Now consider two families $F^{(1)}$ and α of sets in X. (If X was supposed to be a topological space, $F^{(1)}$ would stand for the family of closed and α of open sets in X). Both $F^{(1)}$ and α will be considered fixed. Let $\Sigma=\{\sigma\}$ be some collection of finite families of members of α. The family π consisting of one and only one member of each $\sigma\in\Sigma$, will be called the *trace* of Σ. (π is actually a choice function for Σ). Let for $\sigma=\{A_1, \dots, A_n\}$, $A_i\in\alpha$, $i=1, \dots, n$,

$$\rangle\sigma\langle=\{F\in F^{(1)}:F\cap A_i\neq\emptyset,\ i=1, \dots, n\}.$$

4.1. $\cup\{\rangle\sigma\langle:\sigma\in\Sigma\}=\cap\{\rangle|\pi|\langle:\pi$ *trace of* $\Sigma\}$.

P r o o f. Let $F\in\cup\{\rangle\sigma\langle:\sigma\in\Sigma\}$. Then there is a $\sigma_0\in\Sigma$ such that $F\in\rangle\sigma_0\langle$, what means that $F\cap A\neq\emptyset$ for each $A\in\sigma_0$. Since for each π, $|\pi|\supseteq A$ for some $A\in\sigma_0$, it follows that $F\cap|\pi|\neq\emptyset$ for every π. Hence $F\in\cap\{\rangle|\pi|\langle\}$.

Conversely, let $F\in\cap\{\rangle|\pi|\langle\}$. Suppose that for each σ there is an $A^\sigma\in\sigma$, such that $F\cap A^\sigma=\emptyset$. Take $\pi=\{A^\sigma\}$, then $F\cap|\pi|=\emptyset$ (we use *ACF* here). So for some $\sigma\in\Sigma$, $F\in\rangle\sigma\langle$.

The *notation* $A\overset{F^{(1)}}{=}X$ will mean that

$$F^{(1)}\cap\langle X\setminus A\rangle=\emptyset.$$

For example, if $X=\cup\{X_\zeta:\zeta\in\mathcal{Z}\}$, $F^{(1)}=\Pi$, $A=X_\zeta$ then $A\overset{F^{(1)}}{=}X$ (see 3.8). So if the members of F^1 are "big", the "small" parts may be "equal" to X.

4.2. *If* $\bigcup\{\rangle\sigma\langle : \sigma\in\Sigma\} = F^{(1)}$, *then for each trace* π *of* Σ, $|\pi| \overset{F^{(1)}}{=} X$ *holds.*

Proof. By 4.1, $F^{(1)} = \bigcap\{\rangle|\pi|\langle\}$ that is for each π, $F^{(1)} = \rangle|\pi|\langle$. So for $F\in F^{(1)}$ it follows that $F\cap|\pi|\neq\emptyset$ and so F is not contained in $X\setminus|\pi|$. Hence $|\pi| \overset{F^{(1)}}{=} X$.

4.3. *Let for each* π, $|\pi| \overset{F^{(1)}}{=} X$. *Then* $\bigcup\{\rangle\sigma\langle : \sigma\in\Sigma\} = F^{(1)}$.

Proof. Using 4.1 and having that $\rangle|\pi|\langle = \rangle X\langle$, we get

$$\bigcup\{\rangle\sigma\langle : \sigma\in\Sigma\} = \bigcap\{\rangle|\pi|\langle\} = \bigcap\{\rangle X\langle\} = F^{(1)}.$$

4.4. *A proof of the Tychonoff theorem.* Let $\{X_\zeta : \zeta\in\mathcal{Z}\}$ be a family of compact topological spaces, $\Pi = \Pi\{X_\zeta : \zeta\in\mathcal{Z}\}$ their topological product (see 3.8 again). Let $\{\rangle U\langle\}$ be a cover by subbasic open sets. Then

$$\Pi = \bigcup\{\rangle U\langle\}.$$

Now at least one X_ζ is covered by the sets $\{U\}$, for otherwise there would exist $x_\zeta = X_\zeta$ for every ζ not being contained in any U and $A = \{x_\zeta : \zeta\in\mathcal{Z}\}\notin\bigcup\{U\}$ (Here we use AC).

X_ζ being compact and $X_\zeta\subseteq\bigcup\{U\}$, there exists a finite number of sets $U_1, \ldots, U_n$ such that $U_1\cup\cdots\cup U_n = X_\zeta$. Hence $\Pi = \rangle U_1\langle\cup\cdots\cup\rangle U_n\langle$, since the trace $\pi = \{U_1, \ldots, U_n\}$ is such that $|\pi| \overset{\Pi}{=} X$ and 4.3 applies. The rest of the proof is implied by the Alexander lemma.

Next we prove a consequence of TF:

4.5. *Let* $F^{(1)} = \bigcup\{\rangle\sigma\langle : \sigma\in\Sigma\}$. *If every trace* π *of* Σ *has a finite subfamily* π' *such that* $|\pi'| \overset{F(1)}{=} X$, *then there exist a finite subcollection* Σ' *of* Σ *such that*

$$F^{(1)} = \bigcup\{\rangle\sigma\langle : \sigma\in\Sigma'\}.$$

Proof. Consider each σ as a finite topological space taken with the discrete topology. Then the traces π can be considered as the elements of the topological product $\Pi\{\sigma : \sigma\in\Sigma\} = \Pi$. Each π is the element of $\rangle\pi'\langle$ and $>\pi'<$ is an open set in Π. According to TF, Π is compact and there is a finite subcover of the cover $\{\rangle\pi'\langle\}$,

$$\rangle\pi_1'\langle,\ \rangle\pi_2'\langle,\ \ldots,\ \rangle\pi_n'\langle.$$

Let $\Sigma_0 = \{\sigma : \sigma\cap\pi_i'\neq\emptyset$, for some $i\}$. Take an arbitrary π from Π. Since π belongs to some $\rangle\pi_i'\langle$ and for each $A\in\pi_i'$, $A\in\pi$ ($\pi\cap\{A\} = A$ for $A\in\pi_i'$), the union $|\pi_{\Sigma_0}|$ of those $A\in\pi$ which belong to some $\sigma\in\Sigma_0$ is such that $|\pi_{\Sigma_0}| \overset{F^{(1)}}{=} X$. By 4.3,

$$\bigcup\{\rangle\sigma\langle : \sigma\in\Sigma_0\} = F^{(1)}.$$

Now we will use 4.5 to obtain the Alexander lemma (A), which will be taken in the following form:

Let $\mathcal{S}$ be any family of subsets of X with the property: any subfamily which covers X has itself a finite subfamily which also covers X. Then the family S^* of all finite intersections of members of $\mathcal{S}$ enjoys the same property (see [5]).

4.6. 4.5. *implies* (A).

Proof. Let a subfamily $\mathcal{A}^*$ of $\mathcal{S}^*$ cover X. Each $A^* \in \mathcal{A}^*$ is of the form $A_1 \cap \cdots \cap A_n$, where $A_i \in \mathcal{S}$, and let $\sigma(A^*) = \{A_1, \ldots, A_n\}$. Since $A_i \supseteq A^*$, the traces $\Sigma = \{\sigma(A^*) : A^* \in A^*\}$ cover X so that, by 4.3,

$$\bigcup \{\rangle \sigma(A^*) \langle : A^* \in \mathcal{A}^*\} = P(X).$$

Since for each trace there is a finite subfamily which also covers, applying 4.5, we have

$$\bigcup \{\rangle \sigma(\overset{*}{A_i}) \langle : i = 1, \ldots, n\} = P(X),$$

for some $\overset{*}{A_i} \in \mathcal{A}^*$. Take $x \in X$, then $x \in \rangle \sigma(\overset{*}{A_i}) \langle$ for some i. The last relation means that $x \in \overset{*}{A_i}$ and so

$$X = \overset{*}{A_1} \cup \cdots \cup \overset{*}{A_n}.$$

In proving 4.3, or better 4.1, and 4.5 we used *ACF* and *TF*. Combining that with 4.6 we have.

4.7. *The propositions ACF, TF* 4.5 *and A, are all logically equivalent.*

The question of the logical equivalence of A was raised in [5] and has probably been answered so that we do not insist upon it.

4.8. *If X is compact, then $\lambda(X)$ is compact.*

Proof. Let $\lambda(X) = \bigcup \{\rangle \sigma \langle : \sigma \in \Sigma\}$. Since X is compact all traces of Σ have finite subfamilies which also cover X and 4.5 applies.

Note that there are several proofs of this fact. Let us show how, for X being a T_1-space, 4.8 is directly implied by 4.3. Indeed $x \in \rangle \sigma \langle$ means $x \in \cap \cap \{U : U \in \sigma\}$ and the last sets form an open cover which is reducible to a finite one, say $\sigma_1, \ldots, \sigma_n$. Then the traces of $\Sigma_0 = \{\sigma_1, \ldots, \sigma_n\}$ obviously cover X and 4.3 applies.

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