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REMARK ON SOME RESULTS OF S. PREŠIĆ AND S. ZERVOS

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In his thesis ([1], p. 342—343), S. Zervos has proved a theorem which comprises several results of other autors concerning bounds of moduli of zeros of polynomials. In [2], S. Prešić gave a simple lemma, from which he deduced that theorem directly. The Prešić's lemma can be formulated as follows:

Let E be a nonempty set totally ordered by the relation $\leqslant$ and let the function $g(x_1, \ldots, x_k)$ $(x_1, \ldots, x_k,\ g(x_1, \ldots, x_k) \in E)$ be decreasing with respect to each of its arguments. Then the following implication holds:

$$\xi = g(\xi, \ldots, \xi) \Rightarrow \xi \leqslant \max \{\lambda_1, \ldots, \lambda_k,\ g(\lambda_1, \ldots, \lambda_k)\} \ (\lambda_1, \ldots, \lambda_k \in E).$$

We remark here that this statement can be extended by the following

Lemma. Let the nonempty set E be totally ordered by the relation $\leqslant$ and let the function $g(x_1, \ldots, x_k)$ $(x_1, \ldots, x_k,\ g(x_1, \ldots x) \in E$ be decreasing with respect to each of its arguments. Then:

$$(1) \qquad \xi \leqslant g(\xi, \ldots, \xi) \Rightarrow \xi \leqslant \max \{\lambda_1, \ldots, \lambda_k,\ g(\lambda_1, \ldots, \lambda_k)\} \ (\lambda_1, \ldots, \lambda_k \in E),$$

$$(2) \qquad \xi \geqslant g(\xi, \ldots, \xi) \Rightarrow \xi \geqslant \min \{\lambda_1, \ldots, \lambda_k,\ g(\lambda_1, \ldots, \lambda_k)\} \ (\lambda_1, \ldots, \lambda_k \in E).$$

Hence, in particular,

$$(3) \qquad \xi = g(\xi, \ldots, \xi) \Rightarrow$$
$$\min \{\lambda_1, \ldots, \lambda_k,\ g(\lambda_1, \ldots, \lambda_k)\} \leqslant \xi \leqslant \max \{\lambda_1, \ldots, \lambda_k,\ g(\lambda_1, \ldots, \lambda_k)\}$$
$$(\lambda_1, \ldots, \lambda_k \in E).$$

Proof. Implication (1). Let $\xi \leqslant g(\xi, \ldots, \xi)$ and $\lambda = \max \{\lambda_1, \ldots, \lambda_k\}$, where the elements $\lambda_i\,(l \leqslant i \leqslant k)$ are arbitrarly chosen. If $\xi \leqslant \lambda$, then

$$(4) \qquad \xi \leqslant \max \{\lambda_1, \ldots, \lambda_k,\ g(\lambda_1, \ldots, \lambda_k)\}$$

obviously holds. If $\lambda \leqslant \xi$, then

$$\xi \leqslant g(\xi, \ldots, \xi) \leqslant g(\lambda_1, \ldots, \lambda_k)$$

and (4) holds again.

One gets the implication (2) by applying the above result case where relation $\leqslant$ is replaced by relation $\leqslant \stackrel{\text{def}}{=} \geqslant$; in fact, after this change, every maximum becomes a minimum and the function $g(x_1, \ldots, x_k)$ remains decreasing with respect to each argument.

The last assertion is evident.—

We note that, by application of the proved lemma (in fact of (3)), one can simultaneously obtain the upper bound and the lower bound of the root of equation

$$x^n = a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n \left(a_1, a_2, \ldots, a_n \geqslant 0;\ \sum_{i=1}^{n} a_i > 0\right),$$

given by the mentioned theorem of S. Zervos, while the application of the Prešić's lemma only gives directly the upper bound.

REFERENCES

[1] S. Zervos, *Aspects modernes de la localistion des zéros des polynomes d'une variable*, thèse, Sci. math., Paris, 1960.

[2] S. B. Prešić, *Sur un théorème de S. Zervos*, Publ. Inst. Math. t. 10 (24), 1970 pp. 51—52.