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THE CLOSURE IN THE SPACE OF MIKUSIŃSKI'S OPERATORS

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Class C of continuous complex-valued functions of a non-negative real variable forms a commutative algebra without zero divisors where the product is defined as the finite convolution and the sums and scalar products are defined in the usual way. The quotient field of this algebra is the operator field $\mathbf{K}$ of Mikusiński. In this operator field we have two types of convergence.

First type. A sequence of operators $\{a_n\}$ converges to the operator $a \in \mathbf{K}$ if there exists an element $k \in \mathbf{C}$, $k \neq 0$ such that: 1. $ka_n \in \mathbf{C}$ for every $n \in N$; 2. $ka \in \mathbf{C}$; 3. the sequence $\{ka_n\}$ converges almost uniformly (uniformly on every interval $[O, T]$, $T < \infty$) to ka.

Second type. A sequence of operators $\{a_n\}$ converges to the operator $a \in \mathbf{K}$ if there exists a sequence $\{k_n\}$ belonging to $\mathbf{C}$ such that: 1. The sequence $\{k_n\}$ converges almost uniformly to $k \in \mathbf{C}$, $k \neq 0$; 2. $k_n a_n \in \mathbf{C}$ for every $n \in N$; 3. $ka \in \mathbf{C}$ and 4. the sequence $\{k_n a_n\}$ converges almost uniformly to ka.

Let X be a subset of $\mathbf{K}$. The sequence closure of X, denoted by $\overline{X}$, is the set of such elements $x \in \mathbf{K}$ for which there exists a sequence x_n belonging to X and converging to x. The properties of the sequence closure characterize in a sense the topology introduced by the definition of the limit.

K. Urbanik [2] showed that neither type of convergence satisfies Kuratowski's axiom for the closure: $\overline{\overline{X}} = \overline{X}$.

In this paper we have investigated the following question: Does there exist a fixed number n such that $\overline{X}^{(n+1)} = \overline{X}^{(n)}$? We noted by $\overline{X}^{(n)}$ the n-times iterated operation of closure. The answer was negative for both types of convergence. Furthermore a countable iteration of closure does not close it.

We denote by f or $\{f(t)\}$ the representation of a function $f(t)$ in $\mathbf{K}$. The integral operator is $l = \{1\}$ and its powers $l^k = \left\{\frac{t^{k-1}}{(k-1)!}\right\}$ will be noted by $L(k)$. s is the differential operator and $\exp(-\lambda s)$, $\lambda \geqslant 0$, the translation operator. It is well known that $\exp(-\lambda_1 s)\exp(-\lambda_2 s) = \exp[-(\lambda_1 + \lambda_2)s]$ where λ_1 and λ_2 are positive. Also:

$$(1) \qquad \exp(-\lambda s)\, f = \begin{Bmatrix} 0, & 0 \leqslant t < \lambda \\ f(t-\lambda), & 0 \leqslant \lambda < t \end{Bmatrix}$$

L e m m a 1. *If m and p are different natural numbers, we can not find natural numbers n, k, r and q such that:*

$$\exp(ms)\ L(n)+\exp(ps)\ L(k)=\exp(rs)\ L(q) \tag{2}$$

P r o o f. — Relation (2) can be written in the form:

$$\exp[-(p+r)s]\ L(n)+\exp[-(m+r)s]\ L(k)=\exp[-(m+p)s]\ L(q) \tag{3}$$

We suppose without any restriction that $p<m$. The property of the translation operation expressed by relation (1) shows that $p+r=m+p$ or $m=r$.

Now relation (3) reduces to the form:

$$\exp(-ms)\ L(k)=\exp(-ps)\ [L(q)-L(n)]$$

which is impossible.

L e m m a 2. *The sequence $\exp(\lambda_n s)\ L(n)$ for a bounded sequence λ_n converges to zero in both types of convergence.*

P r o o f. — Let $\lambda_n<M$. We consider the sequence

$$\exp(-Ms)\exp(\lambda_n s)\ L(n)=\exp[-(M-\lambda_n)s]\left\{\frac{t^{n-1}}{(n-1)!}\right\}$$

which belongs to **C**. On every interval $0\leqslant t\leqslant T<\infty$ this sequence converges to zero.

T h e o r e m. *There is no fixed number m such that for every subset $X\subset$ **K**: $\overline{X}^{(m-1)}=\overline{X}^{(m)}$ neither for the first type of convergence nor for the second.*

P r o o f. — Let $\{q(i,n)\}$, for a fixed $i=1,2,\dots,k$ be a sequence of different natural numbers. These k sequences have no common elements.

Let us consider the set $X\subset$ **K** consisting of elements which have the following form:

$$\begin{aligned}L[q(1,p_1)]\exp[q(2p_2)]+L[q(2p_2)]\ \exp[q(3,p_3)]+\dots+\\ +L[q(m,p_m)]\ \exp[q(m+1,p_{m+1})].\end{aligned} \tag{4}$$

Let us remember that $L(k)$ means l^k. According to our lemma 1 every element of X consists purely of m addends. We shall show that any sequence:

$$\begin{aligned}L[q(1,n_1)]\ \exp q(2,n_2)+L[q(2\,n_2)\ \exp[q(3,n_3)]+\dots+\\ +L[q(m,n_m)]\ \exp[q)m+1,n_{m+1})]\equiv A_n\end{aligned} \tag{5}$$

can converge only if the sequences $\{n_2\}$, $\{n_3\},\dots,\{n_{m+1}\}$ consist of a finite number of different elements. Only $\{n_1\}$ can be a sequence which tends to $\to\infty$.

Suppose that A_n converges and has a limit of the second type. Then there exists a sequence $k_n\in$ **C** such that $k_nA_n=F_n\in$ **C**. So we have:

$$\begin{aligned}&k_n\{L[q(1,n_1)]\ \exp\left[-\sum_{k=3}^{m+1}q(k,n_k)\right]+\dots+\\ &+L[q(m,n_m)]\ \exp\left[-\sum_{k=2}^{m}q(k,n_k)\right]\}=\\ &=F_n\exp\left[-\sum_{k=2}^{m+1}q(k,n_k)\right]\end{aligned} \tag{6}$$

The right side of relation (6) is a function which vanishes for $0 \leqslant t < \sum_{k=2}^{m+1} q(k, n_k)$. According to Titchmarsh's theorem, $k_n(t)$ is equal to zero for

$$0 \leqslant t < \sum_{k=2}^{m+1} q(k, n_k) - \gamma, \tag{7}$$

where γ defines the interval $[0, \gamma)$ on which the expression in brackets on the left side of relation (6) vanishes. It is easy to see that γ is equal to the minimum of the numbers

$$\sum_{k=3}^{m+1} q(k, n_k), \ldots, \sum_{k=2}^{m} q(k, n_k).$$

If one of the sequences $\{n_2\}$, $\{n_3\}, \ldots, \{n_{m+1}\}$ tends to infinity, $k_n \to 0$ because the interval (7) tends to the interval $[0, \infty)$. In this case the sequence A_n does not converge. The only possibility is that the sequence $\{n_1\}$ tends to infinity. So by lemma 2 in the set $\overline{X}$, the closure of X, the new elements are of the form:

$$L[q(2, p_2)] \exp[q(3, p_3)] + L[q(3, p_3)] \exp[q(4, p_4)] + \ldots + L[q(m, p_m)] \exp[q(m+1, p_{m+1})]. \tag{8}$$

We see that their form is the same as those in (4) only the number of addends is now $m-1$.

The k—th successive use of the operation of closure, $l \leqslant k \leqslant m-1$, gives new elements of the form (4) but with precisely $m-k$ addends. In other words $\overline{X}^{(k-1)} \neq \overline{X}^{(k)}$. $1 \leqslant k \leqslant m-1$. After the m-th application of the operation of closure the only new element is zero, and $\overline{X}^{(m-1)} \neq \overline{X}^{(m)}$. This proves our theorem.

We can construct a set $Y \subset \mathbf{K}$ such that $\overline{Y}^{(\omega_0)} \neq \overline{Y}^{(k)}$, for every $k \in N$.

Let $\{q(i, n)\}$, $i, n \in N$ be different natural numbers. We shall consider the set Y whose elements are:

$$L[q(1, n_1)] \exp[q(2, n_2)] + \{1\}$$

$$L[q(1, n_1)] \exp[q(2, n_2)] + L[q(2, n_2)] \exp[q(3, n_3)] + \left\{\frac{1}{2}\right\}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$L[q(1, n_1)] \exp[q(2, n_2)] + L[q(2, n_2)] \exp[q(3, n_3)] + \ldots + L[q(k, n_k)] \exp[q(k+1, n_{k+1})] + \left\{\frac{1}{k}\right\}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

where $\{n_i\}$ are natural numbers. In every element of Y, from all addends, only $\left\{\frac{1}{k}\right\}$ belongs to $\mathbf{C}$. The element which does non belong to any $\overline{Y}^{(k)}$, $k \in \mathbf{N}$ is zero. To prove this we use the same method as in proving our theorem.

REFERENCES

[1] J. Mikusiński, *Operational calculus*, Pergamon Press (1957).

[2] K. Urbanik, *Sur la structure non-topologique du corps des opérateurs*, Studia Math. XIV (1954), 243—246.