SOME INEQUALITIES FOR THE GAMMA FUNCTION

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O. Introduction

Books [1] and [2] contain certain amount of inequalities involving the gamma function. Most of them give bounds for the expression \( \frac{\Gamma(x)}{\Gamma(y)} \), where \( x, y \) are positive numbers of a special form, as for example \( x = \frac{n}{2}, y = \frac{n-1}{2} \), where \( n > 2 \) is a positive integer. In the first part of this paper we shall give bounds for the expression \( \frac{\Gamma(x)}{\Gamma(y)} \), where \( x, y \) are arbitrary real numbers greater than 1. The comparison of these bounds with the ones contained in [1] and [2] show that not only are inequalities (1.1) more general in form, but that they are also in some cases sharper.

In the second part of this paper we give new bounds for the expression \( \frac{\Gamma(x)}{\Gamma(y)} \Gamma \left( \frac{x+y}{2} \right) \), which is also treated in [1] and [2], while in Part 3 we give inequalities for some more general expressions.

We have, in fact, taken up a remark given in the Preface of [1], which states that a large number of inequalities involving positive integers hold under weaker conditions than those given in [1]. We have not taken into account the results regarding the gamma function which appeared after the publication of [1] and [2], and shall probably do so in another paper.

The authors are indebted to Prof. D. S. Mitrićović, who has read this paper in manuscript and whose suggestions have influenced the final presentation of the text.

1. Bounds for \( \frac{\Gamma(x)}{\Gamma(y)} \)

1.1. Theorem 1. Let \( x > y > 1 \). Then, we have

\[
\frac{x^{x-1} e^y}{y^{y-1} e^x} \leq \frac{\Gamma(x)}{\Gamma(y)} \leq \frac{x^{x-1/2} e^y}{y^{y-1/2} e^x}.
\]
Proof. Let us prove first the left inequality in (1.1). It is equivalent to
\[ \frac{\Gamma(y)e^y}{y^{y-1}} \leq \frac{\Gamma(x)e^x}{x^{x-1}}, \]
i.e.,
\[ \log \frac{e^y \Gamma(y)}{y^{y-1}} \leq \log \frac{e^x \Gamma(x)}{x^{x-1}}, \]
or
(1.2) \quad y + \log \Gamma(y) - y \log y + \log y < x + \log \Gamma(x) - x \log x + \log x.

Consider the function \( f \) defined by
\[ f(x) = x + \log \Gamma(x) - x \log x + \log x. \]

We have
\[ f'(x) = \frac{\Gamma'(x)}{\Gamma(x)} - \log x + \frac{1}{x}. \]

In virtue of section 3.6.55 from [1], p. 288 or [2], p. 283, we conclude that \( f'(x) > 0 \) for \( x > 1 \), i.e., that \( f \) is an increasing function for \( x > 1 \), which for \( x > y > 1 \) implies inequality (1.2), which is equivalent to the left inequality in (1.1).

Let us now prove the right-hand inequality of (1.1). It is equivalent to
\[ \frac{e^x \Gamma(x)}{x^{x-1}} \leq \frac{e^y \Gamma(y)}{y^{y-1}}. \]
or, after taking logarithms, to
(1.3) \quad x + \log \Gamma(x) - x \log x + \frac{1}{2} \log x < y + \log \Gamma(y) - y \log y + \frac{1}{2} \log y.

For the function \( g \), defined by
\[ g(x) = x + \log \Gamma(x) - x \log x + \frac{1}{2} \log x, \]
we have
\[ g'(x) = \frac{\Gamma'(x)}{\Gamma(x)} - \log x + \frac{1}{2x}. \]

Again by section 3.6.55, ([1], p. 288 or [2], p. 283) we see that \( g'(x) < 0 \) for \( x > 1 \), which for \( x > y > 1 \) implies inequality (1.3), i.e., the right-hand inequality of (1.1).

The theorem is proved.

1.2. The following inequalities were proved by W. Gautschi (see [1], p. 286 or [2], p. 281):
(1.4) \quad n^{1-s} \leq \frac{\Gamma(n+1)}{\Gamma(n+s)} \leq (n+1)^{1-s},

where \( n \) is a positive integer, and \( 0 < s < 1 \).
Setting $x = n + 1$, $y = n + s$ in (1.1) we obtain

\[
\frac{(n + 1)^n}{(n + s)^{n + s - 1} e^{s-1}} \leq \frac{\Gamma (n + 1)}{\Gamma (n + s)} \leq \frac{(n + 1)^{n + 1/2}}{(n + s)^{n + s - 1/2}} e^{s-1}.
\]

1° For $s = 1$, inequalities (1.4) (1.5) coincide, since they become equalities.

2° For $s = \frac{1}{2}$, $n = 1$, the left-hand inequality of (1.5) is weaker than the corresponding inequality of (1.4).

3° For $s = \frac{3}{4}$, $n = 1$, the left-hand inequality of (1.5) is sharper than the corresponding inequality of (1.4).

In other words, the left-hand inequalities of (1.4) and (1.5) cannot be compared to each other.

4° The expressions which appear on the right hand side of inequalities (1.4) and (1.5) were compared by D. V. Slavić on a computer. He showed that for a large number of values for $s$ and $n$ the right-hand side of inequality (1.5) is sharper than the right-hand side of (1.4).

1.3. The following inequality

\[
\sqrt{\frac{2n - 3}{4}} \leq \frac{\Gamma \left( \frac{n}{2} \right)}{\Gamma \left( \frac{n-1}{2} \right)} \leq \sqrt{\frac{(n-1)^2}{2(n-1)}},
\]

which holds for $n \geq 2$ ($n$ is a positive integer), was proved by J. T. Chu (see [1], p. 288 or [2], p. 282).

Putting in (1.1) $x = \frac{n}{2}$, $y = \frac{n-1}{2}$ ($n > 2$), we get

\[
\sqrt{\frac{n^{n-2}}{(n-1)^{n-3} 2e}} \leq \frac{\Gamma \left( \frac{n}{2} \right)}{\Gamma \left( \frac{n-1}{2} \right)} \leq \sqrt{\frac{n^{n-1}}{(n-1)^{n-2} 2e}}.
\]

The computer checkings showed that for a large number of values for $n$ the inequalities (1.6) are sharper than inequalities (1.7).

1.4. For $n = 1, 2, \ldots$ and $0 < r < 1$, Sh. Zimering (see [1], p. 289 or [2], p. 283) obtained the following result

\[
\frac{n^r - (n-1)^r}{r} \geq \frac{\Gamma (n + r)}{n!}.
\]

Putting in (1.4) $r = s$, we obtain

\[
n^{s-1} \geq \frac{\Gamma (n + r)}{\Gamma (n + 1)} = \frac{\Gamma (n + r)}{n!}.
\]
Inequality (1.9) is sharper than (1.8). Indeed, by the Lagrange mean value theorem we have

\[ n^r - (n-1)^r = r \cdot \xi^{r-1} > r n^{r-1} \quad (\xi \in (n-1, n)). \]

**Remark.** The proof that inequality (1.9) is sharper than (1.8) is due to R. R. Janić

Putting in (1.1) \( x = n + 1 \), \( y = n + r \), we get

\[ \frac{(n+r)^{n+r-1}}{(n+1)^n} e^{1-r} \geq \frac{\Gamma(n+r)}{\Gamma(n+1)}. \]

Introduce the following notations:

\[ a = n^{r-1}, \quad b = \frac{(n+r)^{n+r-1}}{(n+1)^n} e^{1-r}, \quad c = \frac{n^r - (n-1)^r}{r}. \]

Some values for the differences \( b-a \), \( c-b \) are listed in the following table:

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**Table 1**

From the above Table we see that inequality (1.1) is sharper than (1.8), but it is weaker than (1.4) for a large number of values of \( n \) and \( r \). However, inequality (1.1) can be sharper than (1.8).

2. **Bounds for** \( \frac{\Gamma(x) \Gamma(y)}{\Gamma\left(\frac{x+y}{2}\right)} \).

2.1. **Theorem 2.** For \( x > 1, y > 1 \), we have

\[
\frac{x^{x} y^{y}}{(x+y)^{x+y}} \leq \frac{\Gamma(x) \Gamma(y)}{\Gamma\left(\frac{x+y}{2}\right)^{2}} \leq \frac{x^{x-1} y^{y}}{(x+y-2)^{x+y-2}}.
\]
Proof. The left-hand inequality (2.1) is equivalent to

\[
(\Gamma \left( \frac{x+y}{2} \right))^2 \leq \frac{\Gamma (x) \Gamma (y)}{x^x y^y}.
\]

(2.2)

For the function \( F \), defined by

\[ F (x) = \frac{\Gamma (x)}{x^x}, \]

we have

\[
\frac{d^2}{dx^2} (\log F (x)) = \frac{d^2}{dx^2} (\log \Gamma (x)) - \frac{1}{x}.
\]

In virtue of section 3.6.55 ([1], p. 288 or [2], p. 283), for \( x > 1 \), we have \((\log F (x))'' > 0\), which means that the function \( x \mapsto \log F (x) \) is convex. Applying the well known Jensen inequality for convex functions to the function \( x \mapsto \log F (x) \), we obtain the left-hand inequality of (2.1).

Let us now prove the right-hand inequality of (2.1). For the function \( G \), defined by

\[ G (x) = \frac{\Gamma (x)}{(x-1)^{x-1}}, \]

we have \( x > 1 \),

\[
\frac{d^2}{dx^2} (\log G (x)) = \frac{d^2}{dx^2} (\log \Gamma (x)) - \frac{1}{x-1} < 0,
\]

(again by 3.6.55, [1], p. 288, or [1], p. 283), which implies

\[
\left( \frac{\Gamma \left( \frac{x+y}{2} \right)}{\frac{x+y-2}{2}} \right)^2 \geq \frac{\Gamma (x) \Gamma (y)}{(x-1)^{x-1} (y-1)^{y-1}}.
\]

The above inequality is equivalent to the right-hand inequality of (2.1). This completes the proof of Theorem 2.

2.2. Inequality

\[
\frac{\Gamma (x) \Gamma (y)}{\Gamma \left( \frac{x+y}{2} \right)} > 1,
\]

(2.3)

was proved by D. Ž. Đoković and P. M. Vasić (see [1], pp. 285—286, or [2], pp. 280—281).

Using the inequality

\[
\left( \sum_{i=1}^{n} x_i \right)^{n} \prod_{i=1}^{n} x_i \leq \prod_{i=1}^{n} x_i^x_i
\]

(see [3] or [4], p. 188), which holds for \( x_i > 1 \), we conclude that the left-hand inequality of (2.1) is sharper than (2.3).
2.3. The following inequality was proved by J. Gurland (see [1], p. 287 or [2], p. 282)
\[
\frac{\Gamma (c-2b) \Gamma (c)}{\Gamma (c-b)^2} \geq \frac{b^2 + c}{c}
\]
and it holds for \(c > 0\) and \(c - 2b > 0\). This inequality for \(c = y, b = \frac{y-x}{2}\) becomes
\[
\frac{\Gamma (x) \Gamma (y)}{\Gamma \left( \frac{x+y}{2} \right)^2} \geq \frac{(y-x)^2 + 4y}{4y}.
\]
(2.4)

The left-hand inequality in (2.1) is in some cases weaker, and in some cases stronger than inequality (2.4). Table 2 shows that as \(y\) increases, inequality (2.1) becomes more and more sharp than (2.4).

2.4. The following inequality, which holds for \(c > 2, c-2b > 0, b \neq 0, b \neq -1\),
\[
\frac{\Gamma (c-2b) \Gamma (c)}{\Gamma (c-b)^2} \geq 1 + \frac{b^2 (c-2)}{(c-b-1)^2}
\]
is due to D. Gokhale (see [1], p. 287, or [2], p. 282).

It can be written in the form
\[
\frac{\Gamma (x) \Gamma (y)}{\Gamma \left( \frac{x+y}{2} \right)^2} \geq 1 + \frac{(y-x)^2 (y-2)}{(x+y-2)^2}.
\]
(2.5)

Though in some cases inequality (2.5) is stronger than the left-hand inequality of (2.1), Table 2 shows that as \(y\) increases, (2.1) becomes more and more sharp than (2.5).

Introduce the following abbreviations:
\[
A = \frac{(y-x)^2 + 4y}{4y}, \quad B = 1 + \frac{(y-x)^2 (y-2)}{(x+y-2)^2}, \quad C = \frac{x^y y^x}{\left( \frac{x+y}{2} \right)^{x+y}}.
\]

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Some inequalities for the gamma function

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**Table 2**

From Table 2 we see that $C>B$ for $y>x+3$. Naturally, $A=B=C$ for $x=y$. We also see that $C>A$ (and $B>A$) for $x>y$.

3. Generalisations of $\frac{\Gamma(x)\Gamma(y)}{\Gamma\left(\frac{x+y}{2}\right)^2}$.

3.1. Since for a convex function $f$ we have

$$f(\frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i}) \leq \frac{\sum_{i=1}^{n} p_i f(x_i)}{\sum_{i=1}^{n} p_i} \tag{3.1}$$

and for a concave function $f$, we have the inequality which is opposite to (3.1), applying (3.1) to the function $x \mapsto \log \frac{\Gamma(x)}{x^x}$ (which is convex) and the opposite inequality to $x \mapsto \log \frac{\Gamma(x)}{(x-1)^{x-1}}$ (which is concave), we obtain the inequalities.

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\[
\prod_{i=1}^{n} (x_i - 1)^{p_i} x_i^{-1} \prod_{i=1}^{n} \Gamma(x_i) p_i \prod_{i=1}^{n} x_i^{p_i x_i} \\
\left( \sum_{i=1}^{n} p_i x_i \right) - \sum_{i=1}^{n} \prod_{i=1}^{n} p_i \prod_{i=1}^{n} \left( \sum_{i=1}^{n} p_i \right) \left( \sum_{i=1}^{n} p_i \right) \left( \sum_{i=1}^{n} p_i x_i \right) \\
\left( \sum_{i=1}^{n} p_i \right) \left( \sum_{i=1}^{n} p_i x_i \right) \\
\left( \sum_{i=1}^{n} p_i \right) \left( \sum_{i=1}^{n} p_i x_i \right)
\]

which generalise (2.1).

3.2. For \( x > 0 \), the following formula holds

\[
\frac{d^k}{dx^k} (\log \Gamma(x)) = (-1)^k \sum_{n=0}^{\infty} \frac{(k-1)!}{(x+n)^k} \quad (k \geq 2)
\]

(see, for example, [5]). This implies that the function \( x \mapsto \log \Gamma(x) \) is convex of order \( 2m - 1 \) (\( m = 1, 2, \ldots \)) and concave of order \( 2m \) (\( m = 1, 2, \ldots \)), in the sense of T. Popoviciu (see [6]). Therefore, we have

\[
\prod_{k=0}^{n} \left( \frac{kx + (n-k)y}{n} \right)^{(-1)^k} \left( \begin{array}{c} n \\ k \end{array} \right) \left\{ \begin{array}{ll} < 1 & \text{for } n = 2m - 1, \\ > 1 & \text{for } n = 2m
\end{array} \right.
\]

where \( m \) is a positive integer.

Inequality (4.1) reduces to (2.3) for \( n = 2 \).

\*
\*
\*

Tables 1 and 2 present short versions of much more extensive tables compiled by D. V. Slavić on an IBM 1130 computer for which the authors wish to express their gratitude.

**REFERENCES**