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SOME INEQUALITIES FOR THE GAMMA FUNCTION

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O. Introduction

Books [1] and [2] contain certain amount of inequalities involving the gamma function. Most of them give bounds for the expression $\frac{\Gamma(x)}{\Gamma(y)}$, where x, y are positive numbers of a special form, as for example $x=\frac{n}{2}$, $y=\frac{n-1}{2}$, where $n \geqslant 2$ is a positive integer. In the first part of this paper we shall give bounds for the expression $\frac{\Gamma(x)}{\Gamma(y)}$, where x, y are arbitrary real numbers greater than 1. The comparison of these bounds with the ones contained in [1] and [2] show that not only are inequalities (1.1) more general in form, but that they are also in some cases sharper.

In the second part of this paper we give new bounds for the expression $\frac{\Gamma(x)\,\Gamma(y)}{\Gamma\left(\frac{x+y}{2}\right)^2}$ which is also treated in [1] and [2], while in Part 3 we give inequalities for some more general expressions.

We have, in fact, taken up a remark given in the Preface of [1], which states that a large number of inequalities involving positive integers hold under weaker conditions than those given in [1]. We have not taken into account the results regarding the gamma function which appeared after the publication of [1] and [2], and shall probably do so in another paper.

The authors are indebted to Prof. D. S. Mitrinović, who has read this paper in manuscript and whose suggestions have influenced the final presentation of the text.

1. Bounds for $\frac{\Gamma(x)}{\Gamma(y)}$

1.1. Theorem 1. *Let $x \geqslant y > 1$. Then, we have*

$$\frac{x^{x-1}\,e^{y}}{y^{y-1}\,e^{x}} \leqslant \frac{\Gamma(x)}{\Gamma(y)} \leqslant \frac{x^{x-\frac{1}{2}}\,e^{y}}{y^{y-\frac{1}{2}}\,e^{x}}. \tag{1.1}$$

Proof. Let us prove first the left inequality in (1.1). It is equivalent to

$$\frac{\Gamma(y)\, e^y}{y^{y-1}} \leqslant \frac{\Gamma(x)\, e^x}{x^{x-1}},$$

i. e.,

$$\log \frac{e^y\, \Gamma(y)}{y^{y-1}} \leqslant \log \frac{e^x\, \Gamma(x)}{x^{x-1}},$$

or

$$y + \log \Gamma(y) - y \log y + \log y \leqslant x + \log \Gamma(x) - x \log x + \log x. \tag{1.2}$$

Consider the function f defined by

$$f(x) = x + \log \Gamma(x) - x \log x + \log x.$$

We have

$$f'(x) = \frac{\Gamma'(x)}{\Gamma(x)} - \log x + \frac{1}{x}.$$

In virtue of section 3.6.55 from [1], p. 288 or [2], p. 283, we conclude that $f'(x) > 0$ for $x > 1$, i. e., that f is an increasing function for $x > 1$, which for $x \geqslant y > 1$ implies inequality (1.2), which is equivalent to the left inequality in (1.1).

Let us now prove the right-hand inequality of (1.1). It is equivalent to

$$\frac{e^x\, \Gamma(x)}{x^{x-\frac{1}{2}}} \leqslant \frac{e^y\, \Gamma(y)}{y^{y-\frac{1}{2}}}.$$

or, after taking logarithms, to

$$x + \log \Gamma(x) - x \log x + \frac{1}{2} \log x \leqslant y \log \Gamma(y) - y \log y + \frac{1}{2} \log y. \tag{1.3}$$

For the function g, defined by

$$g(x) = x + \log \Gamma(x) - x \log x + \frac{1}{2} \log x,$$

we have

$$g'(x) = \frac{\Gamma'(x)}{\Gamma(x)} - \log x + \frac{1}{2x}.$$

Again by section 3.6.55, ([1], p. 288 or [2], p. 283) we see that $g'(x) < 0$ for $x > 1$, which for $x \geqslant y > 1$ implies inequality (1.3), i. e., the right-hand inequality of (1.1).

The theorem is proved.

1.2. The following inequalities were proved by W. Gautschi (see [1], p. 286 or [2], p. 281):

$$n^{1-s} \leqslant \frac{\Gamma(n+1)}{\Gamma(n+s)} \leqslant (n+1)^{1-s}, \tag{1.4}$$

where n is a positive integer, and $0 \leqslant s \leqslant 1$.

Setting $x=n+1$, $y=n+s$ in (1.1) we obtain

(1.5) $$\frac{(n+1)^n}{(n+s)^{n+s-1}}e^{s-1}\leqslant\frac{\Gamma(n+1)}{\Gamma(n+s)}\leqslant\frac{(n+1)^{n+\frac{1}{2}}}{(n+s)^{n+s-\frac{1}{2}}}e^{s-1}.$$

1° For $s=1$, inequalities (1.4) (1.5) coincide, since they become equalities.

2° For $s=\frac{1}{2}$, $n=1$, the left-hand inequality of (1.5) is weaker than the corresponding inequality of (1.4).

3° For $s=\frac{3}{4}$, $n=1$, the left-hand inequality of (1.5) is sharper than the corresponding inequality of (1.4).

In other words, the left-hand inequalities of (1.4) and (1.5) cannot be compared to each other.

4° The expressions which appear on the right hand side of inequalities (1.4) and (1.5) were compared by D. V. Slavić on a computer. He showed that for a large number of values for s and n the right-hand side of inequality (1.5) is sharper than the right-hand side of (1.4).

1.3. The following inequality

(1.6) $$\sqrt{\frac{2n-3}{4}}<\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}<\sqrt{\frac{(n-1)^2}{(2n-1)}},$$

which holds for $n\geqslant 2$ (n is a positive integer), was proved by J. T. Chu (see [1], p. 288 or [2], p. 282).

Putting in (1.1) $x=\frac{n}{2}$, $y=\frac{n-1}{2}$ $(n>2)$, we get

(1.7) $$\sqrt{\frac{n^{n-2}}{(n-1)^{n-3}\,2e}}<\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}<\sqrt{\frac{n^{n-1}}{(n-1)^{n-2}\,2e}}.$$

The computer checkings showed that for a large number of values for n the inequalities (1.6) are sharper than inequalities (1.7).

1.4. For $n=1, 2, \ldots$ and $0<r<1$, Sh. Zimering (see [1], p. 289 or [2], p. 283) obtained the following result

(1.8) $$\frac{n^r-(n-1)^r}{r}\geqslant\frac{\Gamma(n+r)}{n!}.$$

Putting in (1.4) $r=s$, we obtain

(1.9) $$n^{r-1}\geqslant\frac{\Gamma(n+r)}{\Gamma(n+1)}=\frac{\Gamma(n+r)}{n!}.$$

Inequality (1.9) is sharper than (1.8). Indeed, by the Lagrange mean value theorem we have

$$n^r-(n-1)^r=r\,\xi^{r-1}>rn^{r-1} \qquad (\xi\in(n-1,\, n)).$$

R e m a r k. The proof that inequality (1.9) is sharper than (1.8) is due to R. R. Janić

Putting in (1.1) $x=n+1,\ y=n+r$, we get

$$\frac{(n+r)^{n+r-1}}{(n+1)^n}\,e^{1-r}\geqslant\frac{\Gamma(n+r)}{\Gamma(n+1)}.$$

Introduce the following notations:

$$a=n^{r-1},\quad b=\frac{(n+r)^{n+r-1}}{(n+1)^n}\,e^{1-r},\quad c=\frac{n^r-(n-1)^r}{r}.$$

Some values for the differences $b-a$, $c-b$ are listed in the following

n	r	$b-a$	$c-b$	n	r	$b-a$	$c-b$
1	0.1	0.24157885	8.75842116	3	0.7	0.00230463	0.04013582
10	0.1	0·00480357	0·00124861	15	0.7	0.00106489	0.00350631
20	0.1	0.00132562	0.00024220	30	0.7	0.00048475	0.00134412
30	0.1	0.00061968	0.00009806	1	0.8	—0.02265722	0.27265723
1	0.3	0.08932920	2.24400413	2	0.8	—0.00458456	0.06041040
10	0.3	0.00439641	0.00301158	3	0.8	—0.00092874	0.03209163
20	0.3	0·00142644	0.00078599	4	0.8	—0.00018774	0.02096452
30	0.3	0.00072876	0.00037093	15	0.8	0.00053920	0.00344693
1	0.5	0.00963146	0.99036854	30	0.8	0.00028424	0.00142701
10	0.5	0.00330744	0.00502010	1	0.9	—0.01535950	0.12647061
20	0.5	0.00127879	0.00158843	2	0.9	—0.00461588	0.03387843
30	0.5	0.00071703	0.00083031	7	0.9	—0.00012077	0.00633392
1	0.6	—0.01108312	0.67774979	8	0.9	—0.00003848	0.00536430
2	0.6	0.00673880	0.09493052	9	0.9	0.00001281	0.00463994
15	0.6	0.00154186	0.00311782	10	0.9	0.00004558	0.00407982
30	0.6	0.00063263	0.00110476	20	0.9	0.00009835	0.00178937
1	0.7	—0.02147349	0.45004492	30	0.9	0.00008076	0.00112013
2	0.7	—0.00061096	0.08050822				

TABLE 1

From the above Table we see that inequality (1.1) is sharper than (1.8), but it is weaker than (1.4) for a large number of values of n and r. However, inequality (1.1) can be sharper than (1.8).

2. Bounds for $\dfrac{\Gamma(x)\,\Gamma(y)}{\Gamma\left(\dfrac{x+y}{2}\right)^2}$.

2.1. T h e o r e m 2. *For* $x>1,\ y>1$, *we have*

$$\frac{x^x\,y^y}{\left(\dfrac{x+y}{2}\right)^{x+y}}\leqslant\frac{\Gamma(x)\,\Gamma(y)}{\Gamma\left(\dfrac{x+y}{2}\right)^2}\leqslant\frac{x^{x-1}\,y^{y-1}}{\left(\dfrac{x+y-2}{2}\right)^{x+y-2}}. \tag{2.1}$$

Proof. The left-hand inequality (2.1) is equivalent to

$$\left(\frac{\Gamma\left(\frac{x+y}{2}\right)}{\left(\frac{x+y}{2}\right)^{\frac{x+y}{2}}}\right)^2 \leqslant \frac{\Gamma(x)\,\Gamma(y)}{x^x y^y}. \tag{2.2}$$

For the function F, defined by

$$F(x)=\frac{\Gamma(x)}{x^x},$$

we have

$$\frac{d^2}{dx^2}(\log F(x))=\frac{d^2}{dx^2}(\log\Gamma(x))-\frac{1}{x}.$$

In virtue of section 3.6.55 ([1], p. 288 or [2], p. 283), for $x>1$, we have $(\log F(x))''>0$, which means that the function $x\mapsto\log F(x)$ is convex. Applying the well known Jensen inequality for convex functions to the function $x\mapsto\log F(x)$, we obtain the left-hand inequality of (2.1).

Let us now prove the right-hand inequality of (2.1). For the function G, defined by

$$G(x)=\frac{\Gamma(x)}{(x-1)^{x-1}},$$

we have $x>1$,

$$\frac{d^2}{dx^2}(\log G(x))=\frac{d^2}{dx^2}(\log\Gamma(x))-\frac{1}{x-1}<0,$$

(again by 3.6.55, [1], p. 288, or [1], p. 283), which implies

$$\left(\frac{\Gamma\left(\frac{x+y}{2}\right)}{\left(\frac{x+y-2}{2}\right)^{\frac{x+y-2}{2}}}\right)^2 \geqslant \frac{\Gamma(x)\,\Gamma(y)}{(x-1)^{x-1}(y-1)^{y-1}}.$$

The above inequality is equivalent to the right-hand inequality of (2.1). This completes the proof of Theorem 2.

2.2. Inequality

$$\frac{\Gamma(x)\,\Gamma(y)}{\Gamma\left(\frac{x+y}{2}\right)^2}\geqslant 1, \tag{2 3}$$

was proved by D. Ž. Đoković and P. M. Vasić (see [1], pp. 285—286, or [2], pp. 280—281).

Using the inequality

$$\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^{\sum_{i=1}^{n} x_i}\leqslant\prod_{i=1}^{n} x_i^{x_i}$$

(see [3] or [4], p. 188), which holds for $x_i>1$, we conclude that the left-hand inequality of (2.1) is sharper than (2.3).

2.3. The following inequality was proved by J. Gurland (see [1], p. 287 or [2], p. 282)

$$\frac{\Gamma(c-2b)\,\Gamma(c)}{\Gamma(c-b)^2} \geqslant \frac{b^2+c}{c}$$

and it holds for $c>0$ and $c-2b>0$. This inequality for $c=y$, $b=\frac{y-x}{2}$ becomes

$$\frac{\Gamma(x)\,\Gamma(y)}{\Gamma\left(\frac{x+y}{2}\right)^2} \geqslant \frac{(y-x)^2+4y}{4y}. \tag{2.4}$$

The left hand inequality in (2.1) is in some cases weaker, and in some cases stronger than inequality (2.4). Table 2 shows that as y increases, inequality (2.1) becomes more and more sharp than (2.4).

2.4. The following inequality, which holds for $c>2, c-2b>0, b\neq 0, b\neq -1$,

$$\frac{\Gamma(c-2b)\,\Gamma(c)}{\Gamma(c-b)^2} \geqslant 1+\frac{b^2(c-2)}{(c-b-1)^2}$$

is due to D. Gokhale (see [1], p. 287, or [2], p. 282).

It can be written in the form

$$\frac{\Gamma(x)\,\Gamma(y)}{\Gamma\left(\frac{x+y}{2}\right)^2} \geqslant 1+\frac{(y-x)^2(y-2)}{(x+y-2)^2}. \tag{2.5}$$

Though in some cases inequality (2.5) is stronger than the left hand inequality of (2.1), Table 2 shows that as y increases, (2.1) becomes more and more sharp than (2.5).

Introduce the following abbreviations:

$$A=\frac{(y-x)^2+4y}{4y}, \quad B=1+\frac{(y-x)^2(y-2)}{(x+y-2)^2}, \quad C=\frac{x^x\,y^y}{\left(\frac{x+y}{2}\right)^{x+y}}.$$

x	y	$B-A$	$C-A$	$C-B$
1	1	0.00000000	0.00000000	0.00000000
1	2	—0.12500000	0.06018518	0.18518518
1	3	0.66666666	0.35416666	—0.31250000
1	4	1.43750000	1.05893999	—0.37856000
1	5	2.20000000	2.48669411	0.28669410
1	15	9.73333334	1551.44495487	1541.71162176
1	30	20.99166669	25903288.31250000	25903267.31250000
2	1	—1.25000000	—0.06481481	1.18518518
2	5	0.62999999	0.49282507	—0.13717492
2	6	1.11111111	1.18098960	0.06987849
2	15	6.94777778	273.71154570	266.76376807
2	30	17.85777781	2420231.26171875	2420213.40527343
3	1	—2.00000000	—0.31250000	1.68750000
3	2	—0.12500000	—0.01908000	0.10591999
3	6	0.35969387	0.28978689	—0.06990698

x	y	$B-A$	$C-A$	$C-B$
3	7	0.67857142	0.70550311	0.02693168
3	15	4.91250000	75.37120610	70.45870611
3	30	15.16537461	369847.78430175	369832.61889648
4	1	—3.25000000	—0.62856000	2.62143999
4	3	—0.04333333	—0.00902877	0.03430455
4	7	0.23412698	0.19191194	—0.04221503
4	8	0.46000000	0.47308072	0.01308072
4	15	3.42623991	26.69044004	23.26420013
4	30	12.85104168	77015.28396606	77002.43286132
5	1	—5.00000000	—0.71330589	4.28669411
5	4	—0.02168367	—0.00525080	0.01643287
5	8	0.16503099	0.13681548	—0.02821551
5	9	0.33333333	0.34064523	0.00731189
5	15	2.34567901	11.01751812	8.67183911
5	30	10.86145546	20045.71400451	20034.85256195
10	1	—21.25000001	50.53114973	71.78114977
10	9	—0.00355632	—0.00110019	0.00245612
10	13	0.05141287	0.04370638	—0.00770649
10	14	0.11097992	0.11207784	0.00109791
10	30	4.42289936	182.92356908	178.50066971
15	1	—50.00000002	1505.71162176	1555.71162176
15	14	—0.00139623	—0.00046277	0.00093346
15	18	0.02484391	0.02131466	—0.00352925
15	19	0.05509868	0.05544537	0.00034669
15	30	1.53224716	9.91272539	8.38047823
20	1	—91.25000005	37546.58656311	37637.83656311
20	19	—0.00074007	—0.00025336	0.00048670
20	23	0.01460698	0.01259169	—0.00201528
20	24	0.03287981	0.03303095	0.00015113
20	30	0.38194444	0.90342907	0.52148462

TABLE 2

From Table 2 we see that $C>B$ for $y>x+3$. Naturally, $A=B=C$ for $x=y$. We also see that $C\geqslant A$ (and $B\geqslant A$) for $x\geqslant y$.

3. Generalisations of $\dfrac{\Gamma(x)\,\Gamma(y)}{\Gamma\left(\dfrac{x+y}{2}\right)^2}$.

3.1. Since for a convex function f we have

$$f\left(\frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i}\right) \leqslant \frac{\sum_{i=1}^{n} p_i f(x_i)}{\sum_{i=1}^{n} p_i} \tag{3.1}$$

and for a concave function f, we have the inequality which is opposite to (3.1), applying (3.1) to the function $x \mapsto \log \dfrac{\Gamma(x)}{x^x}$ (which is convex) and the opposite inequality to $x \mapsto \log \dfrac{\Gamma(x)}{(x-1)^{x-1}}$ (which is concave), we obtain the inequalities

$$\frac{\prod_{i=1}^{n}(x_i-1)^{p_i x_i-1}}{\left(\frac{\sum_{i=1}^{n}p_i x_i-\sum_{i=1}^{n}p_i}{\sum_{i=1}^{n}p_i}\right)^{\sum_{i=1}^{n}p_i x_i-\sum_{i=1}^{n}p_i}} \geqslant \frac{\prod_{i=1}^{n}\Gamma(x_i)^{p_i}}{\left(\frac{\sum_{i=1}^{n}p_i x_i}{\sum_{i=1}^{n}p_i}\right)^{\sum_{i=1}^{n}p_i}} \geqslant \frac{\prod_{i=1}^{n}x_i^{p_i x_i}}{\left(\frac{\sum_{i=1}^{n}p_i x_i}{\sum_{i=1}^{n}p_i}\right)^{\sum_{i=1}^{n}p_i x_i}}$$

which generalise (2.1).

3.2. For $x>0$, the following formula holds

$$\frac{d^k}{dx^k}(\log\Gamma(x))=(-1)^k\sum_{n=0}^{+\infty}\frac{(k-1)!}{(x+n)^k} \qquad (k\geqslant 2)$$

(see, for example, [5]). This implies that the function $x\mapsto\log\Gamma(x)$ is convex of order $2m-1$ $(m=1, 2, \ldots)$ and concave of order $2m$ $(m=1, 2, \ldots)$, in the sense of T. Popoviciu (see [6]). Therefore, we have

$$\prod_{k=0}^{n}\Gamma\left(\frac{kx+(n-k)y}{n}\right)^{(-1)^k\binom{n}{k}} \begin{cases}\leqslant 1 & \text{for } n=2m-1,\\ \geqslant 1 & \text{for } n=2m\end{cases} \tag{4.1}$$

where m is a positive integer.

Inequality (4.1) reduces to (2.3) for $n=2$.

* * *

Tables 1 and 2 present short versions of much more extensive tables compiled by D. V. Slavić on an IBM 1130 computer for which the authors wish to express their gratitude.

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