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A NEW PROOF OF BELOUSOV'S THEOREM FOR A SPECIAL LAW OF QUASIGROUP OPERATIONS

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1. V. D. Belousov, in the paper [1], proved the following theorem: If four quasigroups $A_i\,(i=1, 2, 3, 4)$ are connected by the general associative law

$$A_1(A_2(x, y), z) = A_3(x, A_4(y, z)),$$

then all A_i are isotopic to one and the same group.

The above theorem is generalized in [2] by Belousov for the case of $2n-2$ binary quasigroup operations.

In the present paper, for a family of $2n-2$ quasigroup operations connected by the following law

$$A_1 A_2 \dots A_{n-1} x_1 x_2 \dots x_n = A_n x_1 A_{n+1} x_2 \dots A_{2n-2} x_{n-1} x_n \tag{1.1}$$

we give a new direct proof of the generalized Belousov's theorem. The type of the considered equations (1.1) is associative one, which means the following:

If $A_i = A_j$ for every $i, j \in \{1, \dots, 2n-2\}$ then the formula (1.1) is the consequence of the associative law.

2. Let $Q(A_i)$ be the family of quasigroup, where operations A_i are defined on the same nonvoid set Q. We are going to prove the following

Theorem 1. *If quasigroups $A_i\,(i=1, \dots, 2n-2)$ are connected by the general law*

$$A_1 A_2 \dots A_{n-1} x_1 x_2 \dots x_n = A_n x_1 A_{n+1} x_2 \dots A_{2n-2} x_{n-1} x_n \tag{1}$$

then all A_i are isotopic to one and the same group.

Proof. Let us introduce notations

$$\begin{array}{ll}
A_{n-1} x_1 x_2 = y_1 & A_{2n-2} x_{n-1} x_n = z_1 \\
A_{n-2} y_1 x_3 = y_2 & A_{2n-3} x_{n-2} z_1 = z_2 \\
A_{n-3} y_2 x_4 = y_3 & A_{2n-4} x_{n-3} z_2 = z_3 \\
\vdots & \vdots \\
A_2 y_{n-3} x_{n-1} = y_{n-2} & \\
A_1 y_{n-2} x_n = y_{n-1} & A_{n+1} x_2 z_{n-3} = z_{n-2}.
\end{array} \tag{2.1}$$

As the operations A_i satisfy the law (1), then from the equation (2.1) follows the equation

(2.2) $$A_n x_1 z_{n-2} = y_{n-1}.$$

Proceeding from $A_i\,(i=1,\dots,2n-2)$ to their left inverse operations, we obtain

(2.3)
$$\begin{array}{ll}
B_{n-1} y_1 x_2 = x_1 & B_{2n-2} z_1 x_n = x_{n-1} \\
B_{n-2} y_2 x_3 = y_1 & B_{2n-3} z_2 z_1 = x_{n-2} \\
B_{n-3} y_3 x_4 = y_2 & B_{2n-4} z_3 z_2 = x_{n-3} \\
\vdots & \vdots \\
B_2 y_{n-2} x_{n-1} = y_{n-3} & B_{n+1} z_{n-2} z_{n-3} = x_2 \\
B_1 y_{n-1} x_n = y_{n-2} &
\end{array}$$

and finally

(2.4) $$B_n y_{n-1} z_{n-2} = x_1$$

where $B_i = {}^{-1}A_i\,(i=1,\dots,2n-2)$, i.e. $B_i xy = z \overset{\text{def}}{\Leftrightarrow} A_i zy = x$. From (2.3) we obtain

$$x_1 = B_{n-1} y_1 x_2 = B_{n-1} B_{n-2} y_2 x_3\; B_{n+1} z_{n-2} z_{n-3} = \dots$$
$$= B_{n-1} B_{n-2} \dots B_2 B_1 y_{n-1} x_n B_{2n-2} z_1 x_n B_{2n-3} z_2 z_1 \dots B_{n+1} z_{n-2} z_{n-3}.$$

Because of (2.4) we have

(2.5) $$B_{n-1} B_{n-2} \dots B_2 B_1 y_{n-1} x_n B_{2n-2} z_1 x_n B_{2n-3} z_2 z_1 \dots B_{n+1} z_{n-2} z_{n-3} = B_n y_{n-1} z_{n-2}$$

Elements y_{n-1}, x_n, $z_i\,(i=1,\dots,n-2)$ are arbitrary elements of the set Q. If we change the above variables as follows: y_{n-1} with x_1, x_n with x_2, z_i with $x_{i+2}\,(i=1,\dots,n-2)$, then (2.5) becomes

(2) $$B_{n-1} B_{n-2} \dots B_2 B_1 x_1 x_2 B_{2n-2} x_3 x_2 B_{2n-3} x_4 x_3 \dots B_{n+1} x_n x_{n-1} = B_n x_1 x_n.$$

From (2) we obtain

(2.6) $$B_{n-2} \dots B_2 B_1 x_1 x_2 B_{2n-2} x_3 x_2 B_{2n-3} x_4 x_3 \dots B_{n+2} x_{n-1} x_{n-2}$$
$$= {}^{-1}B_{n-1} B_n x_1 x_n B_{n+1} x_n x_{n-1}$$

from there we have

$${}^{-1}B_{n-1} B_n x_1 x_n B_{n+1} x_n x_{n-1} = {}^{-1}B_{n-1} B_n x_1 x_n^0 B_{n+1} x_n^0 x_{n-1}$$

where x_n^0 is a fixed element of the set Q, because the left side of equation (2.6) does not contain x_n. This fact allows us to introduce the operation D_{n-3} in the following way:

(2.7) $$B_{n-2} \dots B_2 B_1 x_1 x_2 B_{2n-2} x_3 x_2 B_{2n-3} x_4 x_3 \dots B_{n+2} x_{n-1} x_{n-2} \overset{\text{def}}{=} D_{n-3} x_1 x_{n-1}.$$

The operation D_{n-3} is obviously a quasigroup. Therefore, the equation (2) becomes

(2.8) $$B_{n-1} D_{n-3} x_1 x_{n-1} B_{n+1} x_n x_{n-1} = B_n x_1 x_n.$$

Further, from (2.7) we obtain

(2.9) $$B_{n-3} \dots B_2 B_1 x_1 x_2 B_{2n-2} x_3 x_2 B_{2n-3} x_4 x_3 \dots B_{n+3} x_{n-2} x_{n-3}$$
$$= {}^{-1}B_{n-2} D_{n-3} x_1 x_{n-1} B_{n+2} x_{n-1} x_{n-2},$$

where, according to the same reasoning as before, we can put

(2.10) $B_{n-3} \dots B_2 B_1 x_1 x_2 B_{2n-2} x_3 x_2 B_{2n-3} x_4 x_3 \dots B_{n+3} x_{n-2} x_{n-3} \stackrel{\text{def}}{=} D_{n-4} x_1 x_{n-2}$.

Now the relation (2.7) becomes

(2.11) $$B_{n-2} D_{n-4} x_1 x_{n-2} B_{n+2} x_{n-1} x_{n-2} = D_{n-3} x_1 x_{n-1}.$$

By continuing the procedure, we obtain that the equation (2) is reduced to the following system of $n-2$ equations of general transitivity

(2.12) $$\begin{aligned} &B_2 B_1 x_1 x_2 B_{2n-2} x_3 x_2 = D_1 x_1 x_3 \\ &B_i D_{i-2} x_1 x_i B_{2n-i} x_{i+1} x_i = D_{i-1} x_1 x_{i+1} \qquad (i=3, \dots, n-2) \\ &B_{n-1} D_{n-3} x_1 x_{n-1} B_{n+1} x_n x_{n-1} = B_n x_1 x_n. \end{aligned}$$

By the known theorem ([1] p. 99), we have that the four quasigroups which are connected by any equation from the system (2.12) are isotopic to one and the same group. As every of quasigroups D_k $(k=1, \dots, n-3)$ is in two equations of the system (2.12), so we have that all B_i $(i=1, \dots, 2n-2)$ are isotopic to one and the same group $Q(.)$. From there by simple transformation we have that all A_i $(i=1, \dots, 2n-2)$ are isotopic to the group $Q(.)$.

Biography

[1] В. Д. Белоусов, *Системы квазигрупп*, УМН, XX, вып 1 (121), (1965), 75—146.

[2] В. Д. Белоусов, *Уравновешенные тождества в квазигруппах*, Матем. сб. 70 (112), № 1 (1966), 55—97.