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AFFINE MAPPINGS AND ELLIPTIC FUNCTIONS

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In order to lengthen the short but growing list of known inequalities involving elliptic functions (Cf. [1], [5], [7]) we here compare pairs of these functions in which the arguments are related by an affine mapping. In each case we are able to establish piecewise monotoneity and hence obtain sharp upper and lower bounds for the quotients of these related functions. Comparison of these bounds in some cases gives monotoneity for functions involving elliptic integrals, and we use this fact to derive some inequalities for elliptic integrals. Most of these relations were encountered naturally in the study of some geometric problems in quasiconformal mappings [2].

In the sequel we shall assume that $0<k_1<k_2<1$. For $j=1, 2$ we let $k_j'=(1-k_j^2)^{\frac{1}{2}}$ and $K_j=K(k_j)$, where $K=K(k)$ is the complete elliptic integral defined by

$$K=\int_0^1 [(1-t^2)\ (1-k^2t^2)]^{-\frac{1}{2}}\, \mathrm{dt}.$$

For x real, we let $y=(K_2/K_1)\,x$. We show that the following inequalities hold for all real x:

$$(1)\qquad 1\leq\frac{\operatorname{sn}(y, k_2)}{\operatorname{sn}(x, k_1)}\leq\frac{K_2}{K_1}, \qquad (2)\qquad \frac{k_2'\,K_2}{k_1'\,K_1}\leq\frac{\operatorname{cn}(y, k_2)}{\operatorname{cn}(x, k_1)}\leq 1,$$

$$(3)\qquad \frac{k_2'}{k_1'}\leq\frac{\operatorname{dn}(y,k_2)}{\operatorname{dn}(x,k_1)}\leq 1, \qquad (4)\qquad \frac{K_2}{K_1}\leq\frac{\operatorname{tn}(y, k_2)}{\operatorname{tn}(x, k_1)}\leq\frac{k_1'\,K_1}{k_2'\,K_2},$$

$$(5)\qquad \frac{1+k_1'}{1+k_2'}\leq\frac{\operatorname{sd}(y,k_2)\operatorname{cn}(y,k_2)}{\operatorname{sd}(x,k_1)\operatorname{cn}(x,k_1)}\leq\frac{K_2}{K_1}.$$

In addition, we show that, for $0\leq k<1$,

$$(6)\qquad k'K\leq E\leq (1-k^2/2)\,K,$$

where $E=E(k)$ is the complete elliptic integral

$$E=\int_0^1 [(1-k^2t^2)/(1-t^2)]^{\frac{1}{2}}\, \mathrm{dt}.$$

All bounds in (1) through (5) follow from piecewise monotoneity of the functions by substituting special values of the Jacobian elliptic functions for $x=0$, $K_1/2$, and K_1 ([3], pp. 9, 14; [4], #122.01—.02, #122.10). In some cases it is necessary to apply L'Hôpital's rule, using the differentiation formulas for elliptic functions ([3], p. 9; [4], #731). In view of the periodicity and evenness of these functions ([3], p. 13; [4], #122.00, #122.04) it is sufficient to prove these inequalities for $0\leq x\leq 2K_1$.

We temporarily postpone the proof of (1). To establish (2), we write the infinite product expansion ([4], #909; [6], p. 74)

$$\frac{\operatorname{cn}(y,k_2)}{\operatorname{cn}(x,k_1)}=\prod_{n=1}^{\infty}\varphi_n\frac{A_{2,n}-t}{A_{1,n}-t}\cdot\frac{B_{1,n}+t}{B_{2,n}+t},$$

where $t=\sin^2\alpha$, $\alpha=\pi x/2K_1=\pi y/2K_2$, and

$$A_{j,n}=\frac{(1+q_j^{2n})^2}{4q_j^{2n}},\qquad B_{j,n}=\frac{(1-q_j^{2n-1})^2}{4q_j^{2n-1}},\ j=1,\ 2.$$

Here $q_j=q(k_j)$ is Jacobi's Nome defined by $q=q(k)=\exp(-\pi K'/K)$, where $K'=K(k')$, and φ_n is positive and independent of α. Clearly

$$0<A_{2,n}<A_{1,n},\qquad 0<B_{2,n}<B_{1,n}$$

since $0<q_j<1$, and hence each factor in the infinite product is a positive, strictly decreasing function of t in $0\leq t\leq 1$. From this it follows that the function in (2) is monotone decreasing for $0\leq x\leq K_1$ and monotone increasing for $K_1\leq x\leq 2K_1$, and hence (2) is proved. The proofs for (3) and (4) follow the same pattern.

If the function $\operatorname{sn}(y,k_2)/\operatorname{sn}(x,k_1)$ in (1) is differentiated with respect to x, the numerator of the derivative is

$$\frac{K_2}{K_1}\operatorname{sn}(x,k_1)\ \operatorname{cn}(y,k_2)\ \operatorname{dn}(y,k_2)-\operatorname{sn}(y,k_2)\operatorname{cn}(x,k_1)\ \operatorname{dn}(x,k_1).$$

It follows from (3) and (4) that this expression is nonpositive for $0\leq x\leq K_1$ and nonnegative for $K_1\leq x\leq 2K_1$. Hence (1) is proved.

According to Landen's Transformation ([3], p. 72; [4], #163), we may rewrite the function in (5) in the form

$$\frac{1+k_1'}{1+k_2'}\ \frac{\operatorname{sn}((1+k_2')y,k_2^*)}{\operatorname{sn}((1+k_1')x,k_1^*)},\tag{7}$$

where $k_j^*=(1-k_j')/(1+k_j')$. Since $K^*=K(k^*)=(1+k')K/2$, it follows that

$$(1+k_2')y=(1+k_2')\frac{K_2}{K_1}x=(1+k_1')\frac{K_2^*}{K_1^*}x,$$

and the arguments in the numerator and denominator of (7) are related by an affine mapping in such a way that we can apply (1) to this function. Therefore (7), and hence the function in (5), is monotone decreasing for

$$0\leq(1+k_1')x\leq K(k_1^*)=(1+k_1')K_1/2,$$

that is, for $0\leq x\leq K_1/2$, and monotone increasing for $K_1/2\leq x\leq K_1$. Since replacing x by $x+K_1$ (hence y by $y+K_2$) leaves the function in (5) unchanged ([3], p. 13; [4], #122.03), the behavior of the function for $K_1 \leq x \leq 2K_1$ is clear. Thus (5) is established.

Finally, (4) and (5) imply that $k'K^2$ and $(1+k')K$ are monotone decreasing and increasing functions of k, respectively, for $0 \leq k < 1$. If this information is applied to the derivatives of these functions, with the help of the formula for dK/dk ([3], p. 21; [4], #710.00), we arrive at (6). The inequalities are sharp, equality holding when $k=0$.

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