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A VARIATIONAL PRINCIPLE FOR THE THEORY OF LAMINAR BOUNDARY LAYERS IN INCOMPRESSIBLE FLUIDS

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Abstract

In this note a variational method for solving the problem of flow in laminar boundary-layers of incompressible fluids is presented. For the sake of simplicity, the discussion is limited to the appromixate solutions for the steady plane flow with simple geometries. Two examples are considered in details.

I. Introduction

It is well known fact that the general equations of motion of irreversible processes, in use at the present time, are not derivable from Hamilton's variational principle. In continuum mechanics difficulties arise, even in the case of an ideal fluid, when the Eulerian description is used. However, the use of Hamilton's variational principle as a unifying natural law has excited the imagination of physicists and engineers for a very long time. For example according to Th. De Donder "Laws of nature may always be expected to possess specific variational properties as a consequence of their stability". In addition, when the equations of motion are derivable from a variational principle (Hamilton's principle) a general and systematic approximative procedure for establishing the solution can be developed from a direct study of the variotional integral.

Although, the nature of the connection between the approximative solution and the existence of the variational integral is an intriguing physical problem, there have been several attemps to establish the variational technique in dissipative physics.

Recently, Schechter [1], using the concept of Local Potential Theory (Glansdorff-Prigogine method [2], [3]) has shown that the laminar boundary-layer problems in incompressible fluids can be treated with the help of variational method. The variational method of Schechter has a considerable mechanical significance, because the analitical studies of boundary-layer theory is greatly handicapped by the nonlinearity of the governing equations. In many practical cases the variational approach enables one to overcome the difficulties in various nonlinear problems i. e. offers substantial advantages in some cases.

Although very succesful and useful, the Schechter's variational method in the laminar boundary-layers theory differs from the usual variational principles of ordinary mechanics. According to the Local Potential Theory there are two kinds of physical variables (temperature T, velocity v, pressure p etc.) the so called thermodynamic variables which change during the process of variation on an integral, and the variables of the same type (T_0, v_0, p_0 etc.) evaluated at the stationary state and these quantities are not subject to variation. This dual personality of variables must be maintained until the process of variation is complete. After that setting $T_0=T$, $v_0=v$, $p_0=p$ etc., yields the correct differential equations of the process under consideration. (For more details see ref. [1]).

It is easy to show that the Local Potential Principle is a variational technique which is not subsumed by the classical theory of the calculus of variations. In general, the type of procedure indicated by this principle violates the principles of the classical variational calculus. The justification for this method lies in its success, simplicity and scope [1].

The purpose of this note is to present a new variational principle for laminar boundary-layer theory, without violating the classical variational calculus. The description of the flow field by means of generalized coordinates and partial integration technique enables one to describe boundary layer flow by ordinary differential equations of the Lagrangian type. In the main body of the paper the laminar boundary-layer in incompressible fluids growth along the bodies with simple geometries is treated.

2. Variational principle

We consider here the laminar flow of an incompressible viscous fluid, assuming that only small temperature differences occur so that the influence of temperature on the density ρ and viscosity ν may be neglected.

The governing differential equations of the two dimensional laminar boundary-layer of an incompressible fluid are

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}, \tag{1}$$

$$\frac{\partial p}{\partial y}=0 \tag{2}$$

with the continuity equation for axially symmetric body

$$\frac{\partial (ru)}{\partial x}+\frac{\partial (rv)}{\partial y}=0. \tag{3'}$$

or

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{3''}$$

for the plane case.

We will focus attention to the flow along the bodies with the boundaries of the flow field

$$x=0;\ x=L;\ y=0 \text{ and } y=\infty.$$

The velocity of the fluid flow is specified on all boundaries except on the curve $x=L$, hence the variation in velocity does not vanish at $x=L$.

Differential equations (1) and (2) can be derived from a variational principle:

$$\delta I = \delta \iiint \mathcal{L}\, dx dy\, dt = 0 \tag{4}$$

with the Lagrangian of the form:

$$\mathcal{L} = \left\{ l \left[\frac{1}{2} u \left(\frac{\partial u}{\partial x}\right)^2 + v \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x} \frac{\partial u}{\partial t} + \right.\right.$$

$$\left.\left. + \frac{1}{\rho} \frac{\partial u}{\partial x} \frac{\partial p}{\partial x} + \frac{1}{\rho} \frac{\partial v}{\partial x} \frac{\partial p}{\partial y} \right] - \frac{\nu}{2} \left(\frac{\partial u}{\partial y}\right)^2 \right\} e^{x/l} \tag{5}$$

where l is a parameter which tends to zero after finishing the process of variation. In addition the natural conditions at the boundary $x=L$ must be satisfied for the arbitrary variation of velocity ([1] p. 28):

$$\left. \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial u}{\partial x}\right)} \cdot \delta u \right|_{x=L} = 0 \qquad \left. \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial v}{\partial x}\right)} \cdot \delta v \right|_{x=L} = 0$$

For the Lagrangian (5) after dividing with $e^{x/l}$ one has

$$\left. l \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{\partial u}{\partial t} + \frac{1}{\rho} \frac{\partial p}{\partial x} \right) \delta u \right|_{x=L} = 0 \tag{6}$$

$$\left. l \left(\frac{\partial p}{\partial y}\right) \delta v \right| = 0 \tag{6'}$$

Applying the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial t} \cdot \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial u}{\partial t}\right)} - \frac{\partial}{\partial x} \cdot \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial u}{\partial x}\right)} - \frac{\partial}{\partial y} \cdot \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial u}{\partial y}\right)} = 0$$

$$\frac{\partial \mathcal{L}}{\partial v} - \frac{\partial}{\partial t} \cdot \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial v}{\partial t}\right)} - \frac{\partial}{\partial x} \cdot \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial v}{\partial x}\right)} - \frac{\partial}{\partial y} \cdot \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial v}{\partial y}\right)} = 0$$

one has after dividing with $e^{x/l}$:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + l \left[\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial y} \left(v \frac{\partial u}{\partial x} \right) - \frac{1}{2} \left(\frac{\partial u}{\partial x}\right)^2 \right] =$$

$$= -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} - l \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial t} + \frac{1}{\rho} \frac{\partial p}{\partial x} \right) \tag{7}$$

$$\frac{\partial p}{\partial y} = \rho l \left[\frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y} - \frac{\partial}{\partial x} \left(\frac{1}{\rho} \frac{\partial p}{\partial y} \right) \right] \tag{8}$$

Putting

$$l \to 0 \tag{9}$$

one has immediately the equations (1) and (2) and the conditions (6) and (6)′ are satisfied automatically. Hence, a variational approach to the laminar boundary-layer of an incompressible fluid is established.

Since the variational principle does not involve either the continuity equation or boundary conditions, a method for including the continuity equation and boundary conditions is necessary. The method to be employed here is based on the fact that in many physically important situations it is possible to select the trial solution in such a form that the corresponding boundary conditions and continuity equation are identically satisfied, and explicit use of these conditions is not necessary. The same method has been employed in ref. [1].

It would be very difficult to find a physical meaning for the quantity l introduced in (5). It is interesting to note, that in the heat conduction problem a similar parameter has a clear physical interpretation.

It is well known fact that the governing equation for an intensive non--stationary temperature field is [4], [5];

$$c\tau\frac{\partial^2 T}{\partial t^2}+c\frac{\partial T}{\partial t}=k\nabla^2 T \tag{10}$$

where T is temperature, τ — relaxation time, c — constant heat capacity, k — constant thermal conductivity.

The differential equation (10) may be obtained from the Lagrange's equation for a Lagrangian of the form:

$$\mathscr{L}=\left[\frac{\tau c}{2}\left(\frac{\partial T}{\partial t}\right)^2-\frac{K}{2}\left(\frac{\partial T}{\partial x}\right)^2\right]e^{t/\tau}. \tag{11*}$$

If one wants to study the "classical" case i.e. when the relaxation time τ tends to zero, one obtains a situation similar to that in [5].

3. Applications to the plane steady laminar boundary-layers in incompressible fluids

As an application of the previous theory we shall study the well known problems of the plane steady flow in the laminar boundary-layer of an incompressible fluids. The purpose of following examples is only to demonstrate the feasibility and simplicity of variational method presented here. Needless to say that this method may be applied to more complex problems of the bodies with arbitrary shapes.

Let us consider the action integral:

$$I=\int_0^L\int_0^\infty\left\{l\left[\frac{1}{2}u\left(\frac{\partial u}{\partial x}\right)^2+v\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}+\frac{1}{\rho}\frac{\partial u}{\partial x}\cdot\frac{\partial p}{\partial x}+\frac{1}{\rho}\frac{\partial v}{\partial x}\cdot\frac{\partial p}{\partial y}\right]-\right.$$
$$\left.-\frac{\nu}{2}\left(\frac{\partial u}{\partial y}\right)^2\right\}e^{x/l}\,dx\,dy. \tag{12}$$

In order to get an approximative solution we will introduce a profile of velocities u in the boundary-layer of the form:

$$u=V(x)\cdot\psi\left(\frac{y}{f(x)}\right)+V(x)\cdot\frac{dV}{dx}\Phi\left(\frac{y}{f(x)}\right), \tag{13}$$

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* For the application of the Lagrangian (11) in classical heat conduction see ref. [6].

where $V(x)$ is the velocity on the boundary of the boundary-layer and $f(x)$ is some unknown function by the help of which one can describe the change of velocity of profile across the body under consideration. The functions Φ and ψ must be chosen in a form that satisfies the specific boundary conditions.

The form of the approximation (13) is familar, for many approximative approaches in studying of the flow in the laminar steady boundary layers (see for example ref. [7] and [8]).

For the profile u given by (13), the other component v may be found from the continuity equation.

We will focus our attention on cases when the term $\frac{dV}{dx}$ is negligible, i.e. when the bodies have a simple geometry, close to the flat plates. To be more specific let us consider the velocity profile of the form:

$$u(x, y)=V(x)\cdot\left[1-e^{-\frac{y}{f(x)}}\right]. \tag{14}$$

We will choose the boundary conditions in the form:

$$(15)^{\mathrm{I}} \qquad u=0 \qquad \text{for} \quad y=0$$

$$(15)^{\mathrm{II}} \qquad v=0 \qquad \text{for} \quad y=0$$

$$(15)^{\mathrm{III}} \qquad u=V(x) \qquad \text{for} \quad y\to\infty$$

$$(15)^{\mathrm{IIII}} \qquad u=V(x) \qquad \text{for} \quad x=0.$$

The profile (14) satisfies $(15)^{\mathrm{I}}$ and $(15)^{\mathrm{III}}$ and if we assume that the function $f(x)$ hat the property that

$$(15)^{\mathrm{V}} \qquad f(0)=0,$$

$(15)^{\mathrm{IIII}}$ is satisfied also.

Using (14) one has from the continuity equation $(3)^{\mathrm{II}}$

$$v=\frac{dV}{dx}\left(f-fe^{-\frac{y}{f}}-y\right)+Vf'\left(1-\frac{y}{f}e^{-\frac{y}{f}}-e^{-\frac{y}{f}}\right), \quad \left(f'\equiv\frac{df}{dx}\right) \tag{16}$$

where an arbitrary function $C(x)$ has been specified using $(15)^{\mathrm{II}}$. Employing equation (12) and recalling that ([8] p. 21):

$$\frac{1}{\rho}\frac{\partial p}{\partial x}=-V\frac{dV}{dx}$$

than, upon substituting eqs. (14) and (16) and integrating with respect to y:

$$I=\int_0^L\left\{l\left(VV'^2\cdot fA+V'V^2\frac{25}{36}+\frac{V^3f'^2}{f}\cdot\frac{5}{8\cdot 27}\right)-\right.$$

$$\left.-\frac{\nu}{4}\cdot\frac{V^2}{f}\right\}e^{x/l}dx\equiv\int_0^L\mathscr{L}(x, f, f')\,dx, \tag{17}$$

where prime denotes the derivate with respect to x. The number A has infinite value, but it does not have any influence on following considerations.

The function $f(x)$ in (17) can be selected such that the Euler-Lagrange equation for this reduced variational problem is satisfied. Hence the equation

$$\frac{d}{dx}\cdot\frac{\partial\mathcal{L}}{\partial f'}-\frac{\partial\mathcal{L}}{\partial f}=0$$

in our case gives:

$$\frac{2V^3f'}{f}\cdot\frac{5}{8\cdot 27}+V'V^2\frac{25}{36}-\frac{\nu}{4}\cdot\frac{V^3}{f^2}=l\left\{VV'^2A-\right.$$

(18)
$$\left.-\frac{V^3f'^2}{f^2}\cdot\frac{5}{8\cdot 27}-\frac{\partial}{\partial x}\left(V'V^2\frac{25}{36}+\frac{2V^3f'}{f}\,\frac{5}{8\cdot 27}\right)\right\}$$

where $A=-\left(\frac{1}{3}+\frac{1}{2}\lim_{y\to\infty}\lambda\right)$: $\lambda=\frac{y}{f(x)}$.

It has been said previously that the parameter l has arbitrary structure. In order to avoid uncertainty on the right hand side of the equation (18) when $\lambda\to\infty$ we must provide that the parameter l tends to zero faster than $\lambda\to\infty$, if we put for example $l=e^{-\lambda}$ we will satisfy this requirement.

After finishing this limiting process one gets:

(19)
$$V\frac{d}{dx}\left(\frac{f^2}{2}\right)+30\frac{dV}{dx}\left(\frac{f^2}{2}\right)-\frac{27}{5}\nu=0$$

for $f(0)=0$. The solution of (19) is of the form

(20)
$$f(x)=\sqrt{10{,}8\cdot\nu\cdot V_{(x)}^{-30}\cdot\int V_{(x)}^{29}\,dx}.$$

It follows that the form of the function $f(x)$ depends upon the given velocity $V(x)$.

The function $f(x)$ has a simple physical meaning. According to ref. [8] p. 58, we will introduce the following characteristics of the boundary-layer, using eq. (14)

(a) the displacement thickness

(21)
$$\delta^*=\int_0^\infty\left(1-\frac{u}{V}\right)dy=\int_0^\infty e^{-\frac{y}{f}}\,dy=f(x),$$

(b) momentum thickness:

(22)
$$\delta^{**}=\int_0^\infty\frac{u}{V}\left(1-\frac{u}{V}\right)dy=\int_0^\infty\left(1-e^{-\frac{y}{f}}\right)e^{-\frac{y}{f}}\,dy=\frac{f(x)}{2},$$

(c) local shearing stress acting on the body:

(23)
$$\tau_w=\mu\left(\frac{\partial u}{\partial y}\right)_{y=0}=\mu\,\frac{V(x)}{f(x)}.$$

Using (21), (22) and (23) equation (19) may be written in the form:

(24)
$$\frac{d\delta^{**}}{dx}+\frac{1}{V}\frac{dV}{dx}(2\,\delta^{**}+6{,}5\,\delta^*)=2{,}7\,\frac{\tau_w}{\rho V^2}.$$

A similar equation has been found by T. von Karman [9] using the well known integral method. This equation, the so called first momentum integral is of the form:

$$\frac{d\delta^{**}}{dx}+\frac{1}{V}\frac{dV}{dx}\left(2\,\delta^{**}+\delta^{*}\right)=\frac{\tau_w}{\rho V^2}, \tag{25}$$

and plays the role of a basic equation in the approximative study of the boundary-layer. We will show that there are cases when the approximative solution obtained by help of (24) is more accurate than those of (25).

For the profile given by (14) and using (21), (22), (23) Karman's equation (25) will give the differential equation:

$$V\frac{d}{dx}\cdot\left(\frac{f^2}{2}\right)+8\frac{dV}{dx}\left(\frac{f^2}{2}\right)-2\nu=0$$

the solution of which is

$$f_k(x)=2\sqrt{\nu V_{(x)}^{-8}\cdot\int V_{(x)}^{7}dx} \tag{26}$$

for $f_k(0)=0$.

To be more specific let us consider two cases: (*i*) the laminar steady boundary-layer over a flat plate, (*ii*) the flow which is characterised by the external velocity

$$V(x)=C\cdot x^m$$

where C and m are given constants. (The flow past an infinite wedge of opening angle $\beta\pi=\frac{2m\pi}{m+1}$).

(i) **Flat plate**. For this case the external velocity is of the form

$$V(x)=\text{const.}=U.$$

Hence, from (20) one has

$$f(x)=3{,}28\sqrt{\frac{\mu x}{\rho U}}. \tag{27}$$

The profile of velocity is

$$\frac{u}{U}=1-e^{-0{,}304\,\eta} \tag{28}$$

where

$$\eta=y\sqrt{\frac{\rho U}{\mu x}}.$$

The same problem was solved through the help of numerical integration by Howarth [8] p. 29.

It will be convenient to introduce the local friction coefficient in the form:

$$C_f=\frac{\tau_w}{\frac{1}{2}\rho U^2} \tag{29}$$

Using (23), (28) and $V=U$ one has

(30) $$C_f=\frac{0{,}608}{\sqrt{R_{ex}}},$$

where $R_{ex}=\frac{xU}{\nu}$ is the local Reynolds number.

Applying the solution of the Karman equation (26) one has

(31) $$C_f=\frac{1}{\sqrt{R_{ex}}}.$$

For the same problem the following results are known Howarth and Blasius: $C_f=\frac{0{,}664}{\sqrt{R_{ex}}}$; Nikuradse ref. [10] (experimental): $C_f=\frac{0{,}6575}{\sqrt{R_{ex}}}$; Karman — Polhausen ref. [8]: $C_f=\frac{0{,}686}{\sqrt{R_{ex}}}$; Schechter ref. [1]; $C_f=\frac{0.706}{\sqrt{R_{ex}}}$.

(ii) **The flow past an infinite wedge.** For this case the velocity distribution is of the form

(32) $$V(x)=Cx^m; \qquad m, C=\text{const.}$$

Substituting into (20) and integrating one gets

(33) $$f(x)=\sqrt{10{,}8\,C^{-1}\frac{x^{1-m}}{29\,m+1}}.$$

Using (14), (32) and (33) the velocity distribution is

(34) $$\frac{u}{Cx^m}=1-e^{-K_m\cdot\xi},$$

with

(35) $$K_m=\left(\frac{29\,m+1}{5{,}4\,m+5{,}4}\right)^{1/2},$$

$$\xi=\left(\frac{m+1}{2}\right)^{1/2}\cdot\left(\frac{c}{\nu}x^{m-1}\right)^{1/2}\cdot y.$$

For the same problem the solution (26) obtained using the Karman equation is

(36) $$f_k(x)=2\sqrt{\frac{\nu}{c}\cdot\frac{x^{1-m}}{7\,m+1}}.$$

Table 1 shows a comparison of the velocity distribution given by (34) and corresponding results obtained by HARTREE ([8] p. 68—69).

For $\beta=0{,}1$ $(m=0{,}052)$ the skin friction on the wall is, according to Hartree [8] p. 71:

$$\tau_w=0{,}424\sqrt{\mu\,\rho\,c^3x^{3m-1}}.$$

Using (23), (33) and the same value for m we have

$$\tau_w=0{,}480\sqrt{\nu\,\rho\,c^3x^{3m-1}}.$$

The solution (36) obtained using the Karman equation and (23) yields:

$$\tau_w=0{,}581\sqrt{\mu\,\rho\,c^3x^{3m-1}}.$$

Table 1

ξ \ β	0,1 A	0,1 B	0,3 A	0,3 B	0,6 A	0,6 B	1 A	1 B	1,6 A	1,6 B	2 A	2 B
0,0	0	0	0	0	0	0	0	0	0	0	0	0
0,1	0,0644	0,0582	0,0934	0,076	0,1236	0,0966	0,1535	0,1183	0,1879	0,1441	0,2068	0,1588
0,2	0,1248	0,1154	0,1782	0,1490	0,2319	0,1872	0 2834	0,2266	0,3405	0,2726	0,3709	0,2980
0,3	0,1812	0,1715	0,2550	0,2189	0,3268	0,2719	0,3934	0,3252	0,4644	0,3859	0,5010	0,4186
0,4	0,2340	0,2265	0,3246	0,2858	0,4100	0,3506	0,4865	0,4144	0,5651	0,4849	0,6042	0,5219
0,6	0,3296	0,3328	0,4449	0,4100	0,5468	0,4907	0,6321	0,5662	0,7132	0,6446	0,7510	0,6834
1	0,4865	0,5274	0,6251	0,6190	0.7326	0,7056	0,8111	0,7778	0,8752	0,8432	0,9014	0,8717
1,4	0.6067	0,6907	0,7468	0,7743	0,8423	0,8449	0,9030	0,8968	0,9457	0,9375	0,9610	0,9530
2	0,7364	0,8637	0,8595	0,9151	0,9285	0,9514	0,9643	0,9732	0,9844	0,9871	0,9902	0,9914
2,6	0,8233	0,9537	0,9220	0,9754	0,9676	0,9884	0,9868	0,9946	0,9955	0,9980	0,9975	0,9989
3,2	0,8815	0,9883	0,9567	0,9946	0,9853	0,9979	0,9951	0,9992	0,9987	0,9998	0,9993	0,9999

A — ——— (approximative)

B — ———(Hartree)

Using (21), (33) and (30), for $m=\frac{1}{3}$ we have the displacement thickness

$$\delta^* = 1{,}005\sqrt{\frac{\nu}{c}x^{2/3}}.$$

The Karman integral method gives

$$\delta^* = 1{,}13\sqrt{\frac{\nu}{c}x^{2/3}}.$$

The exact solution is [8] p. 72

$$\delta^* = 0{,}985\sqrt{\frac{\nu}{c}x^{2/3}}.$$

Discussion

In this paper an attempt was made to apply a variational method to the theory of the laminar boundary-layer of incompressible fluids using a Lagrangian. The basic rules of variational calculus are preserved. A characteristic of the present method is that there exists a parameter which tends to zero after the process of variation is complete. The physical meaning of this parameter is not clear.

The approximative method presented here can be applied directly and this approach offers the substantial advantages of being clear cut and simple.

Agreement of the results of the present method with the exact solutions and other results is seen to be quite satisfactory. This fact confirms the physical ground of the variational principle described here.

This method may be also applied to more complex situations of non--Newtonian fluids flow, nonstationary flow etc. An investigation concerning these problems will be reported on elsewhere.

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LIST OF SYMBOLS

x — the distance along the solid surface measured from the stagnation point or the leading edge.

y — the normal drawn from the surface towards the fluid,

u, v —components of velocity in x and y directions

t — time,

ρ — density

r — the radius of curvature of the cross section line of the body,

p — pressure,

ν — kinematic viscosity,

L — characteristic length of the body,

μ — viscosity.

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