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ON REPRODUCTIVE SOLUTIONS OF BOOLEAN EQUATIONS

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The aim of this paper is to show that the Löwenheim and Schöder-Poretski-Itoh theorems on Boolean equations can be subsumed to a more general result on reproductive solutions of Boolean equations. The latter theorem, in its turn, can be deduced from a property of certain functional equations, established by Prešić.

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Let S be an arbitrary set and $C(x_1, \ldots x_n)$ a condition or a system of conditions of an arbitrary nature upon the variables $x_1, \ldots, x_n \in S$. A vector $(x_1, \ldots, x_n) \in S^n$ fulfilling the condition(s) C is called a *solution* (or, *particular solution*) of C. A system of mappings φ_i: $S^n \to S$ $(i=1, \ldots, n)$ is said to define the *general* (or, *parametric*) *solution* of C, if the following two conditions are fulfilled:

(i) for every $(p_1, \ldots, p_n) \in S^n$, the vector $(\varphi_1(p_1, \ldots, p_n), \ldots, \varphi_n(p_1, \ldots, p_n))$ is a solution of C;

and conversely,

(ii) every particular solution $(x_1. \ldots, x_n) \in S^n$ of C can be written in the form $x_i = \varphi_i(p_1, \ldots, p_n)$ $(i=1, \ldots, n)$ for suitably chosen $p_1, \ldots, p_n \in S$.

If condition (ii) is replaced by the stronger property

(iii) every particular solution $(x_1, \ldots, x_n) \in S^n$ of C fulfills $x_i = \varphi_i(x_1, \ldots, x_n)$ $(i=1, \ldots, n)$, then the general solution is said to be *reproductive*.

S. Prešić [6] proved the following

Theorem 0. *Let S be an arbitrary set and φ_i: $S^n \to S$ $(i=1, \ldots. n)$.*

(α) The system of equations

(1) $$x_1 = \varphi_1(x_1, \ldots, x_n) \qquad (i=1, \ldots, n)$$

has the general solution

(2) $$x_i = \varphi_i(p_1, \ldots, p_n) \qquad (i=1, \ldots, n)$$

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if and only if the identities

$$\varphi_i(\varphi_1(x_1, \ldots, x_n), \ldots, \varphi_n(x_1, \ldots, x_n)) = \tag{3}$$
$$= \varphi_i(x_1, \ldots, x_n) \qquad (i=1, \ldots, n)$$

are fulfilled.

(β) *If relations* (3) *are verified. the general solution* (2) *is reproductive.*

The *proof* is very simple and will not be reproduced here. Several interesting applications of this theorem are given in the quoted paper and it is the aim of the present article to add to these applications, new proofs for the theorems of Löwenheim and Schröder-Poretski-Itoh (Theorems 1 and 2, respectively) as well as a common generalization of them (Theorem 3).

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We assume that the reader is familiar with the elements of Boolean calculus and we only recall a few definitions.

Let $<B, \cup, ., ', 0,1>$ be an arbitrary Boolean algebra. A mapping $f: B^n \to B$, is said to be a *Boolean function,* if it can be constructed from variables $x_1, \ldots, x_n$ and constants of B by means of superpositions of the basic operations $\cup, ., '$. An equation (inequality $\leqslant$) involving only Boolean functions of the unknowns $x_1, \ldots, x_n$, is called a *Boolean equation* (*inequality*). As is well known, any system of Boolean equations and/or inequalities can be reduced to a single Boolean equation of the form $f=0$ or, if preferred, of the dual form $f'=1$.

In the sequel, we shall need the following

Lemma 1. *Any Boolean function* f: $B^n \to B$ *satisfies the identity*

$$f(x_1 z \cup y_1 z', x_2 z \cup y_2 z', \ldots, x_n z \cup y_n z') = \tag{4}$$
$$= z f(x_1, \ldots, x_n) \cup z' f(y_1, \ldots, y_n).$$

Proof. For fixed $x_1, \ldots, x_n, y_1, \ldots, y_n \in B$ the function in the left-hand side of (4) becomes a Boolean function $g(z)$ and relation (4) reduces to the well-known identity $g(z) = zg(1) \cup z' g(0)$.

We have now established all prerequisites necessary to prove theorems 1-3 below.

Lemma 2. *Let* $(\xi_1, \ldots, \xi_n) \in B^n$ *be a particular solution of the Boolean equation*

$$f(x_1, \ldots \; x_n) = 0. \tag{5}$$

Then (5) *is equivalent to the system*

$$x_i = \xi_i f(x_i, \ldots, x_n) \cup x_i f'(x_1, \ldots, x_n) \qquad (i=1, \ldots, n).\text{*)} \tag{6}$$

Proof. Setting

$$g_i(x_i, \ldots, x_n) := \xi_i f(x_1, \ldots, x_n) \cup x_i f'(x_1, \ldots x_n) \quad (i=1, \ldots, n), \tag{7}$$

*) $f'(x_1, \ldots, x_n)$ stands for $[f(x_1, \ldots, x_n)]'$.

we have to prove that equation (5) is equivalent to the system

$$x_i = g_i(x_1, \ldots, x_n) \qquad (i = 1, \ldots, n). \tag{8}$$

But (5) implies obviously (6), i.e. (8). Conversely, it follows from Lemma 1 that

$$\begin{aligned} f(g_1, \ldots, g_n) &= f(\xi_1, \ldots, \xi_n) f(x_1, \ldots, x_n) \cup \\ &\cup f(x_1, \ldots, x_n) f'(x_1, \ldots, x_n) = 0, \end{aligned} \tag{9}$$

i.e. (8) implies (5), completing the proof.

Theorem 1 (Löwenheim [3], [4]). *Let* $(\xi_1, \ldots, \xi_n) \in B^n$ *be a particular solution of the Boolean equation* (5). *Then*

$$x_i = \xi_i f(p_1, \ldots, p_n) \cup p_i f'(p_1, \ldots, p_n) \qquad (i = 1, \ldots, n), \tag{10}$$

where $p_1, \ldots, p_n$ *are arbitrary parameters in* B, *define the reproductive solution of* (5).

Proof. It follows from identity (9) that the functions (7) satisfy the identities

$$g_i(g_1, \ldots, g_n) = \xi_i f(g_1, \ldots g_n) \cup g_i f'(g_1, \ldots, g_n) = g_i \tag{11}$$

for $i = 1, \ldots, n$. Hence, by Theorem 0, the system (8) has the reproductive solution $x_i = g_i(p_1, \ldots, p_n)$ $(i = 1, \ldots, n)$, which is precisely (10). But (8) is equivalent to (5), by Lemma 2, thus completing the proof.

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In the sequel we denote by $+$ the symmetric difference $x + y = xy' \cup x'y$ and we shall make use of the group properties of this operation.

Theorem 2 (P. S. Poretski [5], E. Schröder [8], M. Itoh [2]).

(a) *The Boolean equation* (5) *is equivalent to each of the equations*

$$x_i = f(x_1, \ldots. x_n) + x_i \qquad (i = 1, \ldots, n). \tag{12}$$

(b) *Equation* (5) *has the general solution*

$$x_i = f(p_1, \ldots, p_n) + p_i \qquad (i = 1, \ldots, n) \tag{13}$$

if and only if

$$f(\alpha_1, \ldots, \alpha_n) f(\alpha'_1, \ldots, \alpha'_n) = 0 \quad (\forall \alpha_1, \ldots, \alpha_n \in \{0, 1\} \subseteq B). \tag{14}$$

(c) *If relations* (14) *hold, the general solution* (13) *is reproductive.*

Proof. The first statement is trivial.

It implies that equation (5) is also equivalent to the system of n equations (12). Therefore, setting

$$h_i(x_1, \ldots, x_n) := f(x_1, \ldots, x_n) + x_i \qquad (i = 1, \ldots, n), \tag{15}$$

we still have to prove that the system $x_i = h_i(x_1, \ldots, x_n)$ $(i=1, \ldots, n)$ has the general solution $x_i = h_i(p_1, \ldots, p_n)$ $(i=1, \ldots, n)$ if and only if relations (14) hold; and if this is the case, the solution $x_i = h_i(p_1, \ldots, p_n)$ $(i=1, \ldots, n)$ is reproductive.

Comparing this statement to Theorem 0, we see that it suffices to prove the equivalence between the conditions (14) and the identities

$$h_i(h_1, \ldots, h_n) = h_i \qquad (i=1, \ldots, n). \tag{16}$$

But, using Lemma 1, we get the following identities:

$$f(h_1, \ldots, h_n) = f(x_1' f \cup x_1 f', \ldots, x_n' f \cup x_n f') =$$

$$= f(x_1', \ldots, x_n') f(x_1, \ldots, x_n) \cup f(x_1, \ldots, x_n) f'(x_1, \ldots, x_n) =$$

$$= f(x_1', \ldots, x_n') f(x_1, \ldots, x_n),$$

hence

$$h_i(h_1, \ldots, h_n) = f(x_1', \ldots, x_n') f(x_1, \ldots, x_n) + h_i(x_1, \ldots, x_n)$$

$$(i=1, \ldots, n),$$

therefore relations (16) hold if and only if

$$f(x_1', \ldots, x_1') f(x_1, \ldots, x_n) = 0. \tag{17}$$

Now the Löwenheim Verification Theorem [4] states that relation (17) holds identically if and only if conditions (14) are fulfilled, thus completing the proof.

Comments. 1) Equation (5) may not have the parametric solution (13), even if it is consistent. For, as is well-known (see, e. g. [9] or [1]), the necessary and sufficient condition for the consistency of (5) is

$$\prod_{\alpha_1, \ldots, \alpha_n \in \{0, 1\}} f(\alpha_1, \ldots, \alpha_n) = 0, \tag{18}$$

which is a property weaker than (14).

(2) Conditions (14) and (18) are equivalent only for $n := 1$, when both of them reduce to $f(0) f(1) = 0$. In this case the function f can be written $f(x) = ax \cup bx'$, so that $h(x) = xf'(x) \cup x' f(x) = x(a' x \cup b' x') \cup x'(ax \cup bx') = a' x \cup bx'$. Thus we obtain the following

Corollary 1. *The Boolean equation $ax \cup bx' = 0$ is equivalent to the equation $x = a' x \cup bx'$. It has the general solution $x = a' p \cup bp'$ if and only if it is consistent. If this is the case, the above solution is reproductive.*

P. S. Poretski [5] proved that equation $x = ax \cup bx'$ is equivalent to the double inequality $b \leqslant x \leqslant a$, while E. Schröder [8] showed that equation $ax \cup bx' = 0$ is equivalent to the double inequality $b \leqslant x \leqslant a'$. From these results we can immediately deduce the above Corollary 1, which could thus be called the Schröder-Poretski theorem. As a matter of fact, this property has been explicitely stated e. g. by L. Couturat [1].

3) Using the disjunctive canonical form of a Boolean function and the notations $x^0 = x'$, $x^1 = x$, we can give another form to the functions (15), namely

$$h_i(x_1, \ldots, x_n) = x_i' f(x_1, \ldots, x_n) \cup x_i f'(x_1, \ldots, x_n) =$$

$$= x_i' \bigcup_{\alpha_1, \ldots, \alpha_n \in \{0,1\}} f(\alpha_1, \ldots, \alpha_n) x_1^{\alpha_1} \ldots x_n^{\alpha_n} \cup$$

$$\cup x_i \bigcup_{\alpha_1, \ldots, \alpha_n \in \{0,1\}} f'(\alpha_1, \ldots, \alpha_n) x_1^{\alpha_1} \ldots x_n^{\alpha_n} =$$

$$= \bigcup_{\substack{\alpha_1, \ldots, \alpha_n \in \{0,1\} \\ \alpha_i = 0}} f(\alpha_1, \ldots, \alpha_n) x_1^{\alpha_1} \ldots x_n^{\alpha_n} \cup$$

$$\cup \bigcup_{\substack{\alpha_1, \ldots, \alpha_n \in \{0,1\} \\ \alpha_i = 1}} f'(\alpha_1, \ldots, \alpha_n) x_1^{\alpha_1} \ldots x_n^{\alpha_n},$$

for $i = 1, \ldots, n$, that is

$$h_i(x_1, \ldots, x_n) = \bigcup_{\alpha_1, \ldots, \alpha_n \in \{0,1\}} [f'(\alpha_1, \ldots, \alpha_n)]^{\alpha_i} x_1^{\alpha_1} \ldots x_n^{\alpha_n} \quad (i = 1, \ldots, n), \tag{19}$$

which is the direct generalization of the form $h(x) = a'x \cup bx' = f'(1)x \cup f(0)x'$ corresponding to the case $n := 1$. As a matter of fact, M. Itoh [2] considered an equation $f = 1$ and obtained the function h_i in the form corresponding to (19); unlike our proof, Itoh's demonstration is direct and somewhat more technical.

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The above theorems have the following structure in common: a system of n Boolean functions $k_i(x_1, \ldots, x_n)$ $(i = 1, \ldots, n)$ is given, such that, under certain conditions, equation (5) is equivalent to the system

$$x_i = k_i(x_1, \ldots, x_n) \qquad (i = 1, \ldots, n) \tag{20}$$

and has the general reproductive solution

$$x_i = k_i(p_1, \ldots, p_n) \qquad (i = 1, \ldots, n) \tag{21}$$

Notice that the latter property is stronger than the former, i. e.: *if* (21) *is the reproductive solution of* (5), *then* (5) *is equivalent to the system* (20). Thus Theorem 1 is stronger than Lemma 2, while Theorem 2 shows that (5) is always equivalent to (12)*), but may not have the reproductive solution (13).

We shall now generalize Theorems 1 and 2, determining necessary and sufficient conditions in order that formulas (21) be the reproductive solution of equation (5).

To do this, we need the following.

Lemma 3. *Let f, g: $B^n \to B$ be Boolean functions and assume that at least one of the equations $f(x_1, \ldots, x_n) = 0$, $g(x_1, \ldots, x_n) = 0$, is consistent. Then the two equations are equivalent if and only if $f = g$.*

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* In particular they are simultaneously consistent or inconsistent.

Proof. The sufficiency being trivial, we have only to prove the necessity. Thus we assume that equations $f=0$ and $g=0$ are equivalent; it follows from the hypothesis that they are both consistent. Since $f=0$ implies $g=0$, a result of L. Löwenheim [4] (see also S. Rudeanu [7]) states that $g \leq f$ holds identically; similarly $f \leq g$, completing the proof.

Theorem 3. *Let the Boolean equation*

$$f(x_1, \ldots, x_n)=0 \tag{5}$$

be consistent. Then the parametric formulas

$$x_i=k_i(p_1, \ldots, p_n) \qquad (i=1, \ldots, n), \tag{21}$$

define the reproductive solution of equations (5) *if and only if the functions k_i are of the form*

$$k_i(x_1, \ldots, x_n):=f(x_1, \ldots, x_n) z_i(x_1, \ldots, x_n)+x_i \quad (i=1, \ldots, n), \tag{22}$$

where the functions $f, z_1, \ldots, z_n$ satisfy the conditions

$$f(\alpha_1, \ldots, \alpha_n)=f(\alpha_1, \ldots, \alpha_n) \bigcup_{i=1}^{n} z_i(\alpha_1, \ldots, \alpha_n), \tag{23}$$

$$f(\alpha_1, \ldots, \alpha_n) f(\alpha_1+z_1(\alpha_1, \ldots, \alpha_n), \ldots, \alpha_n+z_n(\alpha_1, \ldots, \alpha_n))=0, \tag{24}$$

for every $\alpha_1, \ldots, \alpha_n \in \{0, 1\} \subseteq B$.

Proof. According to the remark before Lemma 3, it is necessary that equation (5) be equivalent to the system

$$x_i=k_i(x_1, \ldots, x_n) \qquad (i=1, \ldots, n). \tag{20}$$

We shall first determine necessary and sufficient conditions for the equivalence between (5) and (20). To these conditions we have to add necessary and sufficient conditions in order that the system (20) has the reproductive solution (21); the complete set of conditions obtained in this way will solve our problem.

Since (20) may be written in the form of the unique equation

$$\bigcup_{i=1}^{n}(k_i(x_1, \ldots, x_n)+x_i)=0, \tag{25}$$

Lemma 3 implies that (5) is equivalent to (20) if and only if the identity

$$f(x_1, \ldots, x_n)=\bigcup_{i=1}^{n}(k_i(x_1, \ldots, x_n)+x_i) \tag{26}$$

holds. Therefore, the inequalities

$$k_i(x_1, \ldots, x_n)+x_i \leq f(x_1, \ldots, x_n) \qquad (i=1, \ldots, n) \tag{27}$$

are identically verified, hence there exist n functions $z_1, \ldots, z_n: B^n \rightarrow B$ such that relations

$$k_i(x_1, \ldots, x_n)+x_i=z_i(x_1, \ldots, x_n) f(x_1, \ldots, x_n) \qquad (i=1, \ldots, n), \tag{28}$$

hold identically.

We have thus proved that the functions k_i are of the form (22), while (26) becomes

$$f(x_1, \ldots, x_n)=f(x_1, \ldots, x_n) \bigcup_{i=1}^{n} z_i(x_1, \ldots, x_n); \tag{29}$$

the identity (29) is thus the necessary and sufficient condition for the equivalence between (5) and (20).

It remains now to find necessary and sufficient conditions in order that the system (20) has the reproductive solution (21). In view of theorem 0, this is the same as finding equivalent conditions for the identities

$$k_i(k_1, \ldots, k_n)=k_i \qquad (i=1, \ldots, n). \tag{30}$$

To facilitate computations, we shall use notations like $X=(x_1, \ldots, x_n)$, $Z=(z_1(x_1, \ldots, x_n), \ldots, z_n(x_1, \ldots, x_n))$, $X+Z=(x_1+z_1(x_1, \ldots, x_n), \ldots, x_n+z_n(x_1, \ldots, x_n))$, etc. Then, taking into account (22) and Lemma 1, we deduce successively the following identities:

$$k_i=x_i' f z_i \cup x_i(f' \cup z_i')=(x_i' z_i \cup x_i z_i')f \cup x_i f' \qquad (i=1, \ldots, n),$$

$$f(K)=f(k_1, \ldots, k_n)=f((x_1+z_1)f \cup x_1 f', \ldots, (x_n+z_n)f \cup x_n f')=$$
$$=f(X+Z)f(X) \cup f(X)f'(X)=f(X+Z)f(X) \qquad (i=1, \ldots, n),$$

$$z_i(K)=z_i(k_1, \ldots, k_n)=z_i((x_1+z_1)f \cup x_1 f', \ldots, (x_n+z_n)f \cup x_n f')=$$
$$=z_i(X+Z)f(X) \cup z_i(X)f'(X) \qquad (i=1, \ldots, n),$$

$$k_i(K)=k_i(k_1, \ldots, k_n)=f(K)z_i(K)+k_i=$$
$$=f(X)f(X+Z)z_i(X+Z)+k_i \qquad (i=1, \ldots, n).$$

Hence the identities (30) are equivalent to

$$f(X)f(X+Z)z_i(X+Z)=0 \qquad (i=1, \ldots, n); \tag{31}$$

but the system (31) is equivalent to the single identity

$$f(X)f(X+Z) \bigcup_{i=1}^{n} z_i(X+Z)=0 \tag{32}$$

which, in view of (29) applied with $X:=X+Z$, reduces to

$$f(X)f(X+Z)=0,$$

or, explicitely,

$$f(x_1, \ldots, x_n)f(x_1+z_1(x_1, \ldots, x_n), \ldots, x_n+z_n(x_1, \ldots, x_n))=0. \tag{33}$$

Summarizing, we have found that identities (22), (29) and (33) are the necessary and sufficient conditions in order that the equation (5) has the reproductive solution (21). In view of Löwenheim's Verification Theorem, (29) and (33) are equivalent to (23) and (24), respectively. This completes the proof.

R e m a r k. It may be easier to verify directly the identities (29) and (33) rather than (23) and (24). See e. g. Comment 1 below.

Comments. 1. Let $(\xi_1, \dots, \xi_n)$ be a particular solution of equation (5) and take

$$z_i(x_1, \dots, x_n) := x_i + \xi_i \qquad (i=1, \dots, n). \tag{34}$$

Now consider the equation

$$\bigcup_{i=1}^{n} (x_i + \xi_i) = 0 \tag{35}$$

and notice that it implies (5), for the unique solution of (35) is $x_i := \xi_i$ $(i=1, \dots, n)$, which satisfies also (5). Therefore, in view of Löwenheim's lemma quoted in our Lemma 3, the inequality

$$f(x_1, \dots, x_n) \leqslant \bigcup_{i=1}^{n} (x_i + \xi_i) \tag{36}$$

holds identically, which means that $f, z_1, \dots, z_n$ fulfill (29). On the other hand,

$$f(x_1 + z_1, \dots, x_n + z_n) = f(\xi_1, \dots, \xi)) = 0, \tag{37}$$

so that identities (33) are fulfilled too.

We are thus in the conditions of the above theorem. For this choice of the functions z_i, the functions k_i become

$$k_i = x_i + (x_i + \xi_i) f = x_i f' + \xi_i f = x_i f' \cup \xi_i f \qquad (i=1, \dots, n),$$

i. e. we get Theorem 1.

2) Take

$$z_i(x_1 . \dots, x_n) := 1 \qquad (i=1, \dots, n). \tag{38}$$

Then relation (23) is verified, (24) reduces to (14), and the functions (22) become (15), so that we get Theorem 2.

3) A necessary and sufficient condition for the reproductive solution of a Boolean equation was first given in [7]. This condition is much more complicated than (though certainly equivalent to) the condition given in the present Theorem 3.

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