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NEW CHARACTERIZATIONS OF THE CUBIC LATTICE GRAPH

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In this paper it is shown that the characterization of the cubic lattice graph, given in [3], holds also for $2 \leqslant n \leqslant 7$ except for $n=4$. Two new variants of characterization of the cubic lattice graph are also given.

We consider only finite undirected graphs without loops and multiple edges.

A cubic lattice graph with characteristic $n (n > 1)$ is a graph whose vertices are all the n^3 ordered triplets of n symbols, with two triplets adjacent if and only if they differ in exactly one coordinate.

Let $d(x, y)$ denote the distance between two vertices x and y, $\Delta(x, y)$ the number of vertices adjacent to both x and y, and $n_2(x)$ the number of vertices at the distance 2 from x.

It is proved in [1] that for $n > 7$ the following properties characterize the cubic lattice graph with the characteristic n:

(b_1) The number of vertices is n^3.

(b_2) The graph is connected and regular of degree $3(n-1)$.

(b_3) If $d(x, y)=1$, then $\Delta(x, y)=n-2$.

(b_4) If $d(x, y)=2$, then $\Delta(x, y)=2$.

(b_5) If $d(x, y)=2$, then there are exactly $n-1$ vertices z, adjacent to x, so that $d(z, y)=3$.

It is shown in [2], that for $n > 7$ the properties (b_1)—(b_4) are sufficient for a characterization and that the property (b_5) is a consequence of (b_1)—(b_4).

R. Laskar gives in [3] a new characterization of a cubic lattice graph G:

(P_1) The number of vertices is n^3.

(P_2) G is connected and regular.

(P_3) $n_2(x)=3(n-1)^2$ for all x in G.

(P_4) The distinct eigenvalues of the adjacency matrix of G are $3n-3$, $2n-3$ $n-3$, -3.

It was proved that a cubic lattice graph has properties (P_1)—(P_4) and that (P_1)—(P_4) imply (b_1)—(b_4). Consequently, this characterization holds for $n > 7$.

It is proved in [4] that characterization with properties (b_1)—(b_5) holds for all $n > 1$, except for $n=4$, in which case only one exceptional graph exists, which is given.

Combining Laskar's [3] and Aigner's [4] results, it can be shown that Laskar's characterization [3] holds for all $n \neq 4$.

Theorem 1. *Graph G is the cubic lattice graph with characteristic n* $(n \neq 4)$, *if and only if it has properties* (P_1), (P_2), (P_3), *and* (P_4).

Proof. In order to prove that this characterization holds for $n \leq 7$, we show that (P_1), (P_2), (P_3), (P_4), (b_2), (b_3), and (b_4) imply (b_5).

In the same way as in [3] from (P_2) and (P_4) we have that the adjacency matrix A of G satisfies the relation

$$(1) \qquad A^3 - A^2(3n-9) + A(2n^2-18n+27) + (6n^2-27n+27)I = 6J,$$

where J is a matrix whose all elements are 1, and I is an identity matrix.

Let $d(x, y)=2$. β_i denotes the number of paths of the lengths i which lead from y to x. We have $\beta_1=0$, $\beta_2=2$ and from (1) $\beta_3=6n-12$ is obtainable. The vertex x is adjacent to two vertices, say u_1 and u_2, at the distance 1 from y. The number of paths from y to x of length 3 which come to x from u_1 or u_2 is $2\cdot(n-2)=2n-4$. Let $v_1, \ldots, v_j$ be vertices adjacent to x which are at the distance 2 from y. Exactly two paths of length 2 lead from y to each of vertices $v_1, \ldots, v_j$. Hence, the number of paths of length 3 from y to x which reach x from vertices $v_1, \ldots, v_j$ is $2j$. We have then $\beta_3=2n-4+2j=$ $=6n-12$ and $j=2n-4$. Consequently, the number of vertices z adjacent to x, which are at the distance 3 from y is $3n-3-(2n-4)-2=n-1$ and this represents (b_5).

The exceptional case for $n=4$, given in [4], in the characterization with (P_1), (P_2), (P_3) and (P_4) is also valid. Namely, the corresponding adjacency matrix has eigenvalues 9, 5, 1 and -3 with multiplicities 1, 8, 27 and 27 respectively which is not in contradiction with (P_4). This completes the proof of Theorem 1.

Remark. The mentioned eigenvalues have been determined by Srbijanka Lazarević.

R. Laskar in [3] remarks that the property (P_3) can be replaced by the following two properties:

(P_3') G is edge-regular, i.e. $\Delta(x, y)=\Delta$ for all x, y so that $d(x, y)=1$.

(P_3'') $\Delta(x, y)$ is even, for all x, y so that $d(x, y)=2$.

It is also conjectured that the property (P_3) is implied by other properties (P_1), (P_2), (P_4).

We shall show that (P_3) can be replaced by the supposition (P_3'''), which is weaker than (P_3''):

(P_3''') $\Delta(x, y)>1$ for all x, y so that $d(x, y)=2$.

Actually, we have:

Theorem 2. *Graph G is the cubic lattice graph with the characteristic $n\,(n\neq 4)$ if and only if it has properties (P_1), (P_2), (P_3'''), (P_4).*

Proof. If G is a cubic lattice graph then it naturally possesses properties (P_1), (P_2), (P_3'''), (P_4).

Conversely, if G has these properties, we shall prove that G has properties (b_1)—(b_5).

The properties (b_1) and (b_2) can be easily deduced starting from (P_1), (P_2) and (P_4). Then we have again (1).

Consider two vertices x and y such that $d(x, y)=1$. Let α_1, α_2, α_3, be the numbers of paths of the lengths 1, 2, 3 respectively which lead from x to y.

Let $\bar{\alpha}_2$ be the mean value and $\alpha_{2\min}$ the minimal value of α_2 with respect to all the pairs x, y such that $d(x, y)=1$. Clearly, $\alpha_{2\min} \leqslant \bar{\alpha}_2$.

It can be easily seen from (1) that all diagonal elements of the matrix A^3 are equal to $3(n-1)(n-2)$. Since G is a regular graph of degree $3n-3$ it is clear that $\bar{\alpha}_2=\frac{3(n-1)(n-2)}{3n-3}=n-2$.

Since $\alpha_1=1$ we have from (1)

$$\alpha_3=\alpha_2(3n-9)-2n^2+18n-21. \tag{2}$$

Consider the vertices x and y for which $\alpha_2=\alpha_{2\min}$. Hence, there are exactly $\alpha_{2\min}$ vertices $z_1, z_2, \ldots$ which are adjacent to both x and y. Each path of length 3 from x to y can be classified as the path of one of the following three types:

1° The path passes twice through the same edge.

2° The path has the form (x, w, z_i, y) where w is a vertex different from y and adjacent to both x and z_i.

3° The path reaches y from a vertex at the distance 2 from x and does not pass twice through the same edge.

The number of paths of the first type is equal to $2\cdot(3n-3)-1=6n-7$. There are at least $\alpha_{2\min}(\alpha_{2\min}-1)$ paths of the second type. The existence of the third type paths is ensured with (P_3'''). Actually, every vertex adjacent to y and at distance 2 from x gives at least one such path. We then have at least $3n-4-\alpha_{2\min}$ third type paths.

Hence, $\alpha_3 \geqslant 6n-7+\alpha_{2\min}(\alpha_{2\min}-1)+3n-4-\alpha_{2\min}$ and together with (2) we have $\alpha_{2\min}^2-\alpha_{2\min}(3n-7)+2n^2-9n+10 \leqslant 0$.

Further, it follows $\alpha_{2\min} \geqslant n-2$, which together with $\alpha_{2\min} \leqslant \bar{\alpha}_2=n-2$ implies that $\alpha_2=n-2$ for all x and y. Thus, G has the property (b_3).

To prove that G has the property (b_4) we suppose on the contrary that (b_4) does not hold for G. Then, there are in G two vertices x and y such that $d(x, y)=2$ and $\Delta(x, y)>2$. Let z be one of the vertices adjacent to both x and y. For the number α_3 of paths of length 3 from x to z we have then $\alpha_3 \geqslant 6n-7+(n-2)(n-3)+2n-2+1=n^2+3n-2$. But, from (2) we have $\alpha_3=n^2+3n-3$, since $\alpha_2=n-2$. This contradiction proves that $\Delta(x, y)=2$ for all x, y such that $d(x, y)=2$.

The property (b_5) may be deduced in the similar way as in Theorem 1. This completes the proof of Theorem 2.

We shall give one more variant of characterisation of the cubic lattice graph. The cubic lattice graph can be expressed as the sum of three complete graphs with n vertices. According to [5] the spectrum of this graph is given by

$$\lambda_f=3n-3-fn, \quad p_f=\binom{3}{f}(n-1)^f, \quad f=0, 1, 2, 3, \tag{3}$$

where p_f is the multiplicity of the eigenvalue λ_f.

Remark. For $n=4$ spectrum (3) is identical with the spectrum of Aigner's exceptional graph. Hence, the cubic lattice graph with characteristic 4 and Aigner's graph represent a pair of nonisomorphic, isospectral regular graphs.

We shall prove that properties (P_1), (P_2), (P_4) can be substituted by the property: G has the spectrum given by (3), so that we have

Theorem 3. *The graph G is cubic lattice graph with characteristic $n\,(n\neq 4)$ if and only if its spectrum is given by* (3) *and it has the property* (P_3''').

Proof. We prove that from (3) it follows that G has properties (P_1), (P_2) and (P_4).

From $\sum_{f=0}^{3} p_f = n^3$ follows (P_1).

From (3) clearly follows (P_4).

Before we prove that (P_2) follows from (3) we prove a lemma. The greatest number r from the spectrum of a graph we call, according to [6], the index of graph. Let q be the mean value of the degrees of vertices i.e. $q=\frac{2p}{m}$ where p is the number of edges and m the number of vertices in a graph.

Lemma. *A graph G is regular if and only if $r=q$.*

Proof of the Lemma. If G is regular then $r=q$. We prove that $r=q$ implies that G is regular.

Theorem 2. of paper [6] contains our Lemma under condition that G is connected. Hence, we suppose that G is not connected. Let $G_1,\ldots,G_s$ be components of G; $r_1,\ldots,r_s$—their indices and $q_1\ldots,q_s$—corresponding mean values of degrees of vertices.

According to [6] we have $q_1\leqslant r_1,\ldots,q_s\leqslant r_s$, Since $r=\max\,(r_1,\ldots,r_s)$ we have $q_1,\ldots,q_s\leqslant r$. If at least one of the quantities $q_1,\ldots,q_s$ were smaller than r, then the inequality $q<r$ would hold, which is impossible, and we get $q_1=\cdots=q_s=q=r$ and $q_1\geqslant r_1,\ldots,q_s\geqslant r_s$. Hence, $q_1=r_1=r,\ldots,q_s=r_s=r$ and we see that all components of G are regular graphs of degree r. G is then also regular, and this completes the proof of the Lemma.

Since the number of vertices and the number of edges of a graph can be determined by its spectrum, we have as a consequence of the Lemma that the graph spectrum provides information about regularity of the graph.

We continue the proof of Theorem 3. Since (3) is the spectrum of the cubic lattice graph we have according to Lemma, that the graph G is regular of degree $3n-3$. Since $3n-3$ is a simple eigenvalue, we have that the adjacency matrix is indecomposible, i.e. that G is connected. In the same way as in [3] then we have that the adjacency matrix A of G satisfies the relation (1). The proof can be further continued in the same way as the proof of Theorem 2.

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