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SOME THEOREMS ON THE FIXED POINT IN LOCALLY CONVEX SPACES

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The intention of this paper is to show that the theorems on the fixed point in locally convex spaces given by A. Deleanu and G. Marinescu [1] can be proved under some less restrictive conditions. Afterwords we shall apply our results on differential equations for Mikusinski's operators [3].

A. Deleanu and G. Marinescu proved the following theorem:

Theorem A. *Let $\mathcal{E}$ be a locally convex space sequentially complete, $\mathcal{A}$ be a saturated family of seminorms defining the topology of $\mathcal{E}$, φ a mapping of $\mathcal{A}$ into $\mathcal{A}$ such that $\varphi[\varphi(\alpha)]=\varphi(\alpha)$ $\forall\,\alpha\in\mathcal{A}$, $\mathcal{M}$ be a subset closed in $\mathcal{E}$ and T be a mapping of $\mathcal{M}$ into $\mathcal{M}$ satisfying the following conditions:*

1. *For every $\alpha\in\mathcal{A}$ there exists $q_\alpha>0$ such that*

$$|Tx-Ty|_\alpha\leqslant q_\alpha\,|x-y|_{\varphi(\alpha)},\quad \forall\, x, y\in\mathcal{M}$$

2. *$q_{\varphi(\alpha)}<1$ for every $\alpha\in\mathcal{A}$.*

Then T has in $\mathcal{M}$ a unique fixed point.

These authors also generalized a theorem of M. A. Krasnoseljski [2] profiting from the theorem *A* and the well known Tychonoff's theorem. In the proof of both theorems they used the supposition $\varphi[\varphi(\alpha)]=\varphi(\alpha)$ which is an unnatural restriction. Moreover, the appication of these theorems in the theory of differential equations of Mikusinski's operators showed that just this restriction must be relaxed.

That is the subject of this paper.

1. Theorems on the fixed point

First we shall give some notations:

Let $\mathcal{E}$ be locally convex vector space sequentially complete;

$I\ I_\alpha$, $\alpha\in\mathcal{J}$ be a saturated family of seminorms;

$\mathcal{M}$ be a subset of $\mathcal{E}$ sequantally complete;

T be a mapping of $\mathcal{M}$ into $\mathcal{M}$.

Theorem 1. *We suppose*:

1. *For every $\alpha\in\mathcal{J}$ there exists $q_\alpha>0$ such that*

$$|Tx-Ty|_\alpha\leqslant q_\alpha\,|x-y|_{\varphi(\alpha)},\quad \forall\, x, y\in\mathcal{M}$$

2. *For every* $\alpha \in \mathcal{J}$ *there exists* n_α *such that for* $n \geqslant n_\alpha$ *we have* $q_{\varphi^n(\alpha)} \leqslant \leqslant q(\alpha) < 1$.

3. *There exists* $x_0 \in \mathcal{M}$ *such that for every* $\alpha \in \mathcal{J}$ *and* $n \geqslant 0$

$$| x_0 - Tx_0 |_{\varphi^n(\alpha)} \leqslant p(\alpha) < \infty, \quad \varphi^\circ(\alpha) = \alpha.$$

Then there exists one and only one solution of the equation $x = Tx$ *which also satisfies the condition:*

4. $| x - x_0 |_{\varphi^n(\alpha)} \leqslant p(\alpha, x) < \infty, \quad n \geqslant 0.$

Proof. — We shall construct the sequence $\{x_n\} \subset \mathcal{M}$ in such a way that $x_i = Tx_{i-1}$ with $x_0 \in \mathcal{M}$. This sequence is a Cauchy sequence:

$$
\begin{aligned}
| x_n - x_{n-1} |_\alpha = | Tx_{n-1} - Tx_{n-2} |_\alpha &\leqslant q_\alpha | x_{n-1} - x_{n-2} |_{\varphi(\alpha)} \\
&\leqslant q_\alpha q_{\varphi(\alpha)} | x_{n-2} - x_{n-3} |_{\varphi^2(\alpha)} \\
&\quad \cdots \cdots \cdots \\
&> \prod_{i=0}^{n-2} q_{\varphi^i(\alpha)} | x_1 - x_0 |_{\varphi^{n-1}(\alpha)} \\
&\leqslant \left(\prod_{i=0}^{n_\alpha} q_{\varphi^i(\alpha)} \right) q^{n-2-n_\alpha}(\alpha) p(\alpha), \quad \text{for} \quad n \geqslant n_\alpha + 2.
\end{aligned} \tag{1}
$$

Using the obtained inequality we have for $n \geqslant n_\alpha + 2$:

$$
\begin{aligned}
| x_{n+m} - x_n |_\alpha &\leqslant \sum_{i=1}^{m} | x_{n+i} - x_{n+i-1} |_\alpha \\
&\leqslant \prod_{i=0}^{n_\alpha} q_{\varphi^i(\alpha)} \sum_{i=1}^{m} q^{n+i-2-n_\alpha}(\alpha) p(\alpha) \\
&\leqslant p(\alpha) \left(\prod_{i=0}^{n_\alpha} q_{\varphi^i(\alpha)} \right) q^{n-1-n_\alpha}(\alpha) \frac{1}{1 - q(\alpha)}.
\end{aligned}
$$

From supposition 2 it follows that x_i is a Cauchy sequence. Let $x \in \mathcal{M}$ be its limit, then x is the required solution:

$$
\begin{aligned}
| x - Tx |_\alpha &\leqslant | x - x_n |_\alpha + | x_n - Tx |_\alpha \\
&\leqslant | x - x_n |_\alpha + q_\alpha | x_{n-1} - x |_{\varphi(\alpha)}.
\end{aligned}
$$

When $n \to \infty$, we have $| x - Tx |_\alpha = 0$ for every $\alpha \in \mathcal{J}$, and consequently $x = Tx$. Now we shall show that x also satisfies condition 4.

From relation (1) it follows for $k \geqslant n_\alpha$

$$
\begin{aligned}
| x_n - x_{n-1} |_{\varphi^k(\alpha)} &\leqslant \prod_{i=0}^{n-2} q_{\varphi^{k+i}(\alpha)} | x_1 - x_0 |_{\varphi^{n+k-1}(\alpha)} \\
&\leqslant q^{n-1}(\alpha) p(\alpha)
\end{aligned}
$$

and

$$|x_n - x_0|_{\varphi^k(\alpha)} \leqslant \frac{1}{1-q(\alpha)} p(\alpha).$$

When $n \to \infty$, we have:

$$|x - x_0|_{\varphi^k(\alpha)} \leqslant \frac{1}{1-q(\alpha)} p(\alpha), \quad k \geqslant n_\alpha.$$

Now without difficulty one can show that condition 4 is satisfied.

Finally, we shall prove the uniqueness of the solution in $\mathcal{M}$ which also satisfies condition 4.

Let on the contrary, x and y be two solutions of the equation $Tx = x$ then:

$$|x-y|_\alpha = |Tx - Ty|_\alpha \leqslant q_\alpha |x-y|_{\varphi(\alpha)}$$

$$\leqslant \left(\prod_{i=0}^{n_\alpha} q_{\varphi^i(\alpha)}\right) q^{n-n_\alpha}(\alpha) |x-y|_{\varphi^{n+1}(\alpha)}$$

$$\leqslant \left(\prod_{i=0}^{n_\alpha} q_{\varphi^i(\alpha)}\right) q^{-n_\alpha}(x) [p(\alpha, x) + p(\alpha, y)] q^n(\alpha).$$

When $n \to \infty$, it follows that $|x-y|_\alpha = 0$ for every $\alpha \in \mathcal{J}$, and consequently $x = y$.

Theorem 2. *We suppose*:

1. *For every* $\alpha \in \mathcal{J}$ *and* $k \in \mathcal{N}$ *there exists* $q_\alpha(k) > 0$ *such that*:

$$|T^k x - T^k y|_\alpha \leqslant q_\alpha(k) |x-y|_{\varphi(\alpha,k)}, \quad \forall x, y \in \mathcal{M}$$

and $\sum_{k \geqslant 1} q_\alpha(k) < \infty$.

2. *For every* $\alpha \in \mathcal{J}$ *and* $x, y \in \mathcal{M}$ *there exists* $0 \leqslant p_\alpha(x, y) < \infty$ *such that*:

$$|x-y|_{\varphi(\alpha,k)} \leqslant p_\alpha(x, y), \quad k \geqslant 1.$$

Then there exists one and only one solution of the equation $x = Tx$ *in* $\mathcal{M}$.

Proof. — Let us construct the sequence $\{x_i\} \subset \mathcal{M}$ which satisfies the relation $x_n = Tx_{n-1}$ starting from an element $x_0 \in \mathcal{M}$. This sequence is a Cauchy sequence:

First we have:

$$|x_n - x_{n-1}|_\alpha = |T^{n-1} x_1 - T^{n-1} x_0|_\alpha$$

$$\leqslant q_\alpha(n-1) |x_1 - x_0|_{\varphi(\alpha, n-1)}$$

$$\leqslant q_\alpha(n-1) p_\alpha(x_1, x_0)$$

and

$$|x_{n+m} - x_n|_\alpha \leqslant \sum_{i=1}^{m} |x_{n+i} - x_{n+i-1}|_\alpha$$

$$\leqslant p_\alpha(x_1, x_0) \sum_{i=n}^{n+m-1} q_\alpha(i).$$

We know that the series $\sum_{k\geqslant 1} q_\alpha(k)$ is convergent, consequently $\sum_{k=1}^{n} q_\alpha(k)$ is a Cauchy sequence. By the last relation the sequence $\{x_i\}$ also is a Cauchy sequence. Let $x\in\mathcal{M}$ be the limit of the sequence $\{x_i\}$, then x is the required solution:

$$|Tx-x|_\alpha \leqslant |Tx-x_{n+1}|_\alpha + |x_{n+1}-x|_\alpha$$

$$\leqslant q_\alpha(1)\,|x-x_n|_{\varphi(\alpha,1)} + |x_{n+1}-x|_\alpha.$$

Since $|Tx-x|_\alpha \to 0$ as $n\to\infty$ for every $\alpha\in\mathcal{J}$ we have $Tx=x$.

Now we shall prove that this solution x is unique. Conversely, assume that x and y from $\mathcal{M}$ are two solutions which satisfy condition 2, then

$$|x-y|_\alpha = |T^k x - T^k y|_\alpha \leqslant q_\alpha(k)\,|x-y|_{\varphi(\alpha,k)}$$

$$\leqslant q_\alpha(k)\, p_\alpha(x,y).$$

Since $q_\alpha(k)\to 0$ for $k\to\infty$, $|x-y|_\alpha=0$ must be true for every $\alpha\in\mathcal{J}$, consequently $x=y$.

A different kind of these theorems is a generalization of a theorem of Krasnoseljski.

Theorem 3. *Let $\mathcal{F}$ be a closed and convex subset of the topological, Hausdorff locally convex, complete space $\mathcal{E}$ and S, T two mappings of $\mathcal{F}$ into $\mathcal{E}$ satisfying the following conditions:*

1. *For every $x, y\in\mathcal{F}$, $Tx+Sy\in\mathcal{F}$,*

2. *T satisfies conditions 1 and 2 of theorem 1 for $\mathcal{M}=\mathcal{F}$,*

3. *S is continuous and $S\mathcal{F}$ a relativelly compact set,*

4. *For every $\alpha\in\mathcal{J}$, there exist $m_\alpha>0$ and $\beta_\alpha\in\mathcal{J}$ such that: $|x|_{\varphi^k(\alpha)}\leqslant m_\alpha |x|_{\beta_\alpha}$, $\forall x\in\mathcal{E}$ and $k=0, 1, 2, \ldots$*

Then there exists at least one point $x_0\in\mathcal{F}$ such that $Sx_0+Tx_0=x_0$.

Proof. — Let x be a fixed element of $\mathcal{F}$. The mapping: $y\to Ty+Sx$ has the unique fixed point in $\mathcal{F}$. Indeed, since T satisfies conditions 1 and 2 of theorem 1, the same conditions are satisfied by the mapping $y\to Ty+Sx$ in $\mathcal{F}$. Also, by condition 4 of this theorem, conditions 3 and 4 of theorem 1 are satisfied. This also implies the uniqueness of the fixed point $y=Ty+Sx$. To obtain this fixed point, we can start from the sequence y_n which is constructed in such a way that $y_n=Ty_{n-1}+Sx$. The first element of this sequence, y_0, may be any element from $\mathcal{F}$. Consequently it does not depend of x. Now we can define the mapping R which maps $\mathcal{F}$ into $\mathcal{F}$ and to every $x\in\mathcal{F}$ corresponds a fixed point of the mapping $y\to Ty+Sx$, namely $Rx=TRx+Sx$. We shall show that the mapping R is continuous on $\mathcal{F}$.

By proving theorem 1 we have established the inequality:

$$|x-x_0|_{\varphi^k(\alpha)} \leqslant \frac{1}{1-q(\alpha)}\,p(\alpha), \quad k\geqslant n_\alpha \tag{2}$$

where x is the fixed point of the mapping T, considered in theorem 1, and x_0 is the first element of the sequence $\{x_n\}$ which converges on x. $p(\alpha)$ is introduced by the inequality:

$$|x_0 - Tx_0|_{\varphi^n(\alpha)} \leqslant p(\alpha) < \infty$$

of supposition 3., theorem 1.

We will apply this to our mapping: $y \to Ty + Sx$. By definition, Rx is the unique fixed point of this mapping, and can be obtained as the limit of the sequence $\{y_n\}$, $y_n = Ty_{n-1} + Sx$, where it does not matter which the first element is. For the first element y_0 let us choose just Rx_0. By supposition 4 of our theorem for $p(\alpha)$ we can take $p(\alpha) = m_\alpha |Rx_0 - TRx_0 - Sx|_{\beta(\alpha)}$. Then the inequality (2) in this case is:

$$|Rx - Rx_0|_{\varphi^k(\alpha)} \leqslant \frac{m_\alpha |y_0 - Ty_0 - Sx|_{\beta(\alpha)}}{1 - q(\alpha)} = \frac{m_\alpha |TRx_0 + Sx_0 - TRx_0 - Sx|_{\beta(\alpha)}}{1 - q(\alpha)}$$

$$\leqslant \frac{m_\alpha |Sx - Sx_0|_{\beta(\alpha)}}{1 - q(\alpha)}$$

because $Rx_0 = TRx_0 + Sx_0$.

Now it is easy to prove that R is continuous:

$$|Rx - Rx_0|_\alpha \leqslant q_\alpha |Rx - Rx_0|_{\varphi(\alpha)} + |Sx - Sx_0|_\alpha$$

$$\leqslant q_\alpha \{q_{\varphi(\alpha)} |Rx - Rx_0|_{\varphi^2(\alpha)} + |Sx - Sx_0|_{\varphi(\alpha)}\} + |Sx - Sx_0|_\alpha$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\leqslant q_\alpha q_{\varphi(\alpha)} \cdots q_{\varphi^{n_\alpha - 1}(\alpha)} |Rx - Rx_0|_{\varphi^{n_\alpha}(\alpha)} +$$

$$+ \sum_{i=0}^{n_\alpha - 1} \left(\prod_{\nu=0}^{i-1} q_{\varphi^\nu(\alpha)} \right) |Sx - Sx_0|_{\varphi^i(\alpha)}$$

$$\leqslant \prod_{i=0}^{n_\alpha - 1} q_{\varphi^i(\alpha)} \frac{m_\alpha}{1 - q(\alpha)} |Sx - Sx_0|_{\beta(\alpha)} +$$

$$+ \sum_{i=0}^{n_\alpha - 1} \left(\prod_{\nu=0}^{i=1} q_{\varphi^\nu(\alpha)} \right) |Sx - Sx_0|_{\varphi^i(\alpha)}$$

by the supposition 4 of our theorem we have:

$$\leqslant m_\alpha |Sx - Sx_0|_{\beta(\alpha)} \left\{ \prod_{i=0}^{n_\alpha - 1} q_{\varphi^i(\alpha)} \frac{1}{1 - q(\alpha)} + \sum_{i=0}^{n_\alpha - 1} \prod_{\nu=0}^{i-1} q_{\varphi^\nu(\alpha)} \right\}.$$

The continuity of R is so a consequence of the continuity of S.

We shall prove that $R\mathcal{F}$ is precompact; ($R\mathcal{F}$ is precompact if its completion $\overline{R\mathcal{F}}$ is compact). The necessary and sufficient condition for the precompactness of a set $\mathcal{M}$ is that for every neighbourhood $\mathcal{U}$ of zero, there exists a finite number of points $x_i \in \mathcal{M}$, $i=1, 2, \ldots, n$, such that $\mathcal{M} \subset \bigcup_{i=1}^{n} (x_i + \mathcal{U})$. Let $\mathcal{U}$ be: $\mathcal{U} = \{y : |y|_\alpha < \varepsilon,\ y \in \mathcal{E}\}$. The set $S\mathcal{F}$ is by supposition 3 relatively compact, i.e. $\overline{S\mathcal{F}}$ is compact. We know that the space $\mathcal{E}$ is complete. It follows that $S\mathcal{F}$ is precompact.

Let $\mathcal{W}$ be the neighbourhood of zero of the form:

$$\mathcal{W} = \left\{x : |x|_{\beta(\alpha)} < \frac{\varepsilon}{m_\alpha}\left[\frac{1}{1-q(\alpha)} \prod_{i=0}^{n_\alpha-1} q_{\varphi^i(\alpha)} + \sum_{i=0}^{n_\alpha-1} \prod_{\nu=0}^{i-1} q_{\varphi^\nu(\alpha)}\right]^{-1}\right\}.$$

Then there exists a finite number of points $x_i \in \mathcal{F}$ such that $S\mathcal{F} \subset \bigcup_{i=1}^{n} (Sx_i + \mathcal{W})$. This means that for every $x \in \mathcal{F}$ there exists $i \in 1, 2, \ldots, n$ such that

$$|Sx - Sx_i|_{\beta(\alpha)} < \frac{\varepsilon}{m_\alpha}\left[\frac{1}{1-q(\alpha)} \prod_{i=0}^{n_\alpha-1} q_{\varphi^i(\alpha)} + \sum_{i=0}^{n_\alpha-1} \prod_{\nu=0}^{i-1} q_{\varphi^\nu(\alpha)}\right]^{-1}.$$

Whence, as in (3) $|Rx - Rx_i|_\alpha < \varepsilon$, i.e. $R\mathcal{F} \subset \bigcup_{i=1}^{n} (Rx_i + \mathcal{U})$ which had to be proved.

We know now that $R\mathcal{F}$ is precompact. On account of the completeness of $\mathcal{E}$, $\overline{R\mathcal{F}}$ is compact. Let $\mathcal{K}$ be the closed and convex envelope of $\overline{R\mathcal{F}}$. Since $\mathcal{E}$ is complete, hence $\mathcal{K}$ is compact and we have $R\mathcal{F} \subset \mathcal{K} \subset \mathcal{F}$. Consequently $R\mathcal{K} \subset \mathcal{K}$. So, the set $\mathcal{K}$ and the restriction of R over $\mathcal{K}$ satisfy the conditions of Tychonoff's theorem [7]. That means that there exists $x_0 \in \mathcal{K}$ such that $Rx_0 = x_0$, i.e. $x_0 = Tx_0 + Sx_0$.

2. Application in the theory of operator differential equations

We shall show the application of theorem 2 on differential equations in the field $\mathcal{K}$ of Mikusinski's operators [3] in the case when the conditions of the cited theorem of A. Deleanu and G. Marinescu are not satisfied.

Let $\mathcal{C}$ be the commutative algebra of complex-valued functions defined and continuous on the interval $[0, \infty]$. The sums and scalar products are defined in the usual way and the product is defined as the finite convolution $\left(fg = \left(\int_0^t f(t-u)\, g(u)\, du\right)\right)$. $\mathcal{C}$ is an integral domain under convolution and its field extension is the field $\mathcal{K}$. In $\mathcal{K}$ the limit, differentiation and integration are defined. The field $\mathcal{K}$ consists of „convolution quotients" $\frac{f}{g}$ where $f, g \in \mathcal{C}$ and $g \neq 0$.

We shall let $f = \{f(t)\}$ denote the representation of $f(t)$ in $\mathcal{C}$, s the differential operator, l the integral operator and I the unit element, $s^\circ = I$.

Furthermore, we denote by $F_p(t) = t^{-p-1}\,\Phi(-p, -\sigma; -t^{-\sigma})$ $0 < \sigma < 1$, $p \geqslant 0$, $F_0 \equiv F$, where Φ is the known function of E. M. Wright [8]. This function can be written in the form

$$F_p(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} z^p \exp(tz - z^\sigma)\, dz, \quad t \geqslant 0.$$

This function is continuous for $t \geqslant 0$. Its properties which will be used here are:

1. $|F_p(t)| \leqslant Q h^{\frac{p+1}{\sigma}} (p+1)^{\frac{p+1}{\sigma}}$, Q and h are constant

2. $s^\beta F_q\left(2^{\frac{k}{\sigma}} t\right) = 2^{\frac{k\beta}{\sigma}} F_{q+\beta}\left(2^{\frac{k}{\sigma}} t\right)$

3. $F_q\left(2^{\frac{k}{\sigma}} t\right) = 2^{\frac{q+k+2}{\sigma}} F_q\left(2^{\frac{k+1}{\sigma}} t\right) * F_0\left(2^{\frac{k+1}{\sigma}} t\right)$.

For the function F_p we introduce the followging notations:

$$F_{p,k} = F_p\left(2^{\frac{k}{\sigma}} t\right) \quad \text{and} \quad F \equiv F_{0,0}.$$

Let $\mathcal{J}$ be the interval $[0, \wedge]$ and $\mathcal{R}^+$ the set of non-negative real numbers. Every element $\omega(\lambda, t)$ from the set $\mathcal{C}(\mathcal{J} \times \mathcal{R}^+)$ (continuous functions on $\mathcal{J} \times \mathcal{R}^+$) defines a mapping of $\mathcal{J}$ into $\mathcal{C}(\mathcal{R}^+) : \lambda \to \omega(\lambda) = \{\omega(\lambda, t)\}$. The set of all these mappings is noted by $\tilde{\mathcal{C}}$. In $\tilde{\mathcal{C}}$ a family of seminorms is defined:

$$\nu_k(\omega(\lambda)) = \operatorname*{Max}_{(\lambda, t) \in D_k} |\omega(\lambda, t)|$$

where D_k: $O \leqslant \lambda \leqslant \wedge$, $O \leqslant t \leqslant k$.

$\tilde{\mathcal{C}}$ is complete.

Let us consider the mappings of the interval $\mathcal{J}$ into the field $\mathcal{K}$ which are of the form $\frac{\omega(\lambda)}{F}$ and for every k there exists a $\omega_k(\lambda) \in \tilde{\mathcal{C}}$ such that $\frac{\omega(\lambda)}{F} = \frac{\omega_k(\lambda)}{F_{0,k}}$, $\lambda \in \mathcal{J}$. The set of all these mappings is noted by $\mathcal{C}^*(\lambda)$. This set is not empty. The set $\tilde{\mathcal{C}}$ above all belongs to it.

In $\mathcal{C}^*(\lambda)$ a family of seminorms is also defined. For $\eta(\lambda) \in \mathcal{C}^*(\lambda)$ we have:

$$\|\eta(\lambda)\|_{k,m} = \operatorname*{Max}_{(\lambda, t) \in D_m} |F_{0,k}\, \eta(\lambda)| = \nu_m(F_{0,k}\, \eta(\lambda)).$$

If $\eta(\lambda) \in \tilde{\mathcal{C}}$, then

$$\|\eta(\lambda)\|_{k,m} \leqslant m\, \nu_m(F_{0,k})\, \nu_m(\eta(\lambda)).$$

For every fixed k this family of seminorms is saturated; that is a consequence of the theorem of Titchmarsh.

The set $\mathcal{C}^*(\lambda)$ is sequentially complete. Since it has a countable family of seminorms it is also complete.

To show this, let $\eta_n(\lambda)$ be a Cauchy sequence of $\mathcal{C}^*(\lambda)$ i.e.

$$\|\eta_n(\lambda) - \eta_{n+p}(\lambda)\|_{k,m} = \nu_m(F_{0,k}\, \eta_n(\lambda) - F_{0,k}\, \eta_{n+p}(\lambda)) \to 0, \quad n \to \infty.$$

In this case $F_{0,k}\, \eta_n(\lambda)$ is a Cauchy sequence in $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{C}}$ is complete. Let $\omega_k(\lambda) \in \tilde{\mathcal{C}}$ be its limit. For a fixed k the limit of the sequence $\eta_n(\lambda)$ is

$\frac{\omega_k(\lambda)}{F_{0,k}}$. We shall show that $\frac{\omega_k(\lambda)}{F_{0,k}}$ belongs to the set $\mathcal{C}^*(\lambda)$, i.e. $\omega_k(\lambda)F_{0,p} = \omega_p(\lambda)F_{0,k}$.

We know that $\eta_n(\lambda) \in \mathcal{C}^*(\lambda)$, hence $\eta_n(\lambda) = \frac{y_{n,k}(\lambda)}{F_{0,k}}$, $y_{n,k}(\lambda) \in \tilde{\mathcal{C}}$ for every k and n natural numbers. Since $\eta_n(\lambda) \to \frac{\omega_k(\lambda)}{F_{0,k}}$ we have:

$$\left\| \eta_n(\lambda) - \frac{\omega_k(\lambda)}{F_{0,k}} \right\|_{k,m} = \nu_m(Y_{n,k}(\lambda) - \omega_k(\lambda)) \to 0$$

for every m and k as natural numbers. Using this fact we can show that:

$$\begin{aligned} \| \omega_k(\lambda)F_{0,p} - \omega_p(\lambda)F_{0,k} \|_{k,m} &\leq \| \omega_k(\lambda)F_{0,p} - F_{0,p}F_{0,k}\eta_n(\lambda) \|_{k,m} + \\ &+ \| F_{0,p}F_{0,k}\eta_n(\lambda) - \omega_p(\lambda)F_{0,k} \|_{k,m} \\ &\leq m\,\nu_m(F_{0,p}F_{0,k})\,\nu_m(\omega_k(\lambda) - y_{n,k}(\lambda)) + \\ &+ m\,\nu_m(F_{0,k}^2)\,\nu_m(\omega_p(\lambda) - y_{n,p}(\lambda)) \to 0 \end{aligned}$$

when $n \to \infty$ and k, m, p are natural numbers.

Let us consider the differential equation

$$x'(\lambda) = s^\beta \omega(\lambda)x(\lambda), \quad x(0) = I. \tag{4}$$

where $\omega(\lambda) \in \tilde{\mathcal{C}}$. This equation is equivalent to the integral equation:

$$x(\lambda) = I + \int_0^\lambda s^\beta \omega(u)x(u)\,du \tag{5}$$

We shall apply theorem 2 to equations (4) and (5) respectively and we shall see that it has its unique solution when $\beta < 2$ [4].

For the space E we shall take the space $\mathcal{C}^*(\lambda)$ and for $\mathcal{A}$ the set $[\mathcal{N} \cup \{0\}]^2$.

Let $\mathcal{M}^*$ be the set whose elements are the finite sums $\Sigma s^{\beta i}\omega_i(\lambda)$, $\omega_i(\lambda) \in \tilde{\mathcal{C}}$ and $|\omega_i(\lambda, t)| \leq \frac{\lambda^i t^{i-1} M_m^i}{i!\,(i-1)!}$ where $\nu_m(\omega(\lambda)) \leq M_m$.

The operator T is in our case

$$Tx = I + \int_0^\lambda s^\beta \omega(u)x(u)\,du$$

and it maps $\mathcal{M}^*$ into $\mathcal{M}^*$, which is easy to control. This operator is also continuous, because:

$$\begin{aligned} \| Tx - Ty \|_{k,m} &= \nu_m\left(2^{\frac{\beta k}{\sigma}} F_{\beta,k} \int_0^\lambda \omega(u)[x(u) - y(u)]\,du\right) \\ &\leq \wedge M_m m\,\nu_m\left(2^{\frac{\beta k+\beta+k+2}{\sigma}} F_{\beta,k+1}\right)\nu_m(F_{0,k+1}[x(\lambda) - y(\lambda)]) \\ &\leq m \wedge M_m 2^{\frac{\beta k+k+\beta+2}{\sigma}} \nu_m(F_{\beta,k+1})\,\| x(\lambda) - y(\lambda) \|_{k+1,m} \end{aligned}$$

Let us complete the set $\mathcal{M}^*$ with the limits of Cauchy sequences and we shall obtain the set $\mathcal{M}$. These limits exists because $C^*(\lambda)$ is sequentially complete.

We shall show that condition 1 of theorem 2 is satisfied. Let T_h be the homogeneous part of the mapping T, then

$$\| T^k x - T^k y \|_{p,m} = \| T_h^k (x-y) \|_{p,m}$$

$$= \| \int_0^{\lambda} \omega(\lambda_1)\, d\lambda_1 \int_0^{\lambda_1} \omega(\lambda_2)\, d\lambda_2 \cdots \int_0^{\lambda_{k-1}} \omega(\lambda_k))\, s^{\beta k} [x(\lambda_k) - y(\lambda_k)]\, d\lambda_k \|_{p,m}$$

(6) $$= \nu_m \Big(2^{\frac{p\beta k}{\sigma}} F_{\beta k, p} \int_0^{\lambda} \omega(\lambda_1)\, d\lambda_1 \cdots \int_0^{\lambda_{k-1}} \omega(\lambda_k) [x(\lambda_k) - y(\lambda_k)]\, d\lambda_k \Big)$$

$$\leqslant 2^{\frac{p\beta k + \beta k + p + 2}{\sigma}} m\, \nu_m \Big(F_{\beta k, p+1} \int_0^{\lambda} \omega(\lambda_1)\, d\lambda_1 \cdots \int_0^{\lambda_{k-1}} \omega(\lambda_k)\, d\lambda_k \Big) \times$$

$$\times\, \nu_m (F_{0, p+1} [x(\lambda) - y(\lambda)]) = q_{p,m}(k) \| x - y \|_{m, p+1}$$

It is now necessary to prove that

$$q_{p,m}(k) = 2^{\frac{p\beta k + \beta k + p + 2}{\sigma}} m\, \nu_m \Big(F_{\beta k, p+1} \int_0^{\lambda} \omega(\lambda_1)\, d\lambda_1 \cdots \int_0^{\lambda_{k-1}} \omega(\lambda_k)\, d\lambda_k \Big)$$

has the role of $q_\alpha(k)$ from theorem 2.

Utilizing property 1 of the function $F_p(t)$ we have

$$q_{p,m}(k) \leqslant m\, 2^{\frac{p\beta k + \beta k + p + 2}{\sigma}} Q h^{\frac{\beta k + 1}{\sigma}} (\beta k + 1)^{\frac{\beta k + 1}{\sigma}} \frac{M_m^k \wedge^k T^k}{k!\; k!} = 0 \Big(H^k k^{\left(\frac{\beta}{\sigma} - 2\right) k} \Big)$$

When $\beta < 2$, we can choose σ, $0 < \sigma < 1$, in such a way that $2 - \dfrac{\beta}{\sigma} > 0$, and condition 1 of theorem 2 is satisfied.

With regard to condition 2, it is certainly satisfied, because $\varphi(\alpha, k)$ in our case does not depend on k, one can see it from the inequality (6).

The same theorem is also applicable to a system

(7) $$x_i'(\lambda) = \sum_{j=1}^{m} (a_{i,j}(\lambda))\, x_j(\lambda), \quad i = 1, 2, \ldots, m, \quad x_i(0) = I$$

which is equivalent to the system of integral equations

(8) $$x_i(\lambda) = I + \int_0^{\lambda} \sum_{j=1}^{m} (a_{i,j}(u)\, x_j(u)\, du, \quad i = 1, 2, \ldots, m$$

or in the vector form

(9) $$\xi(\lambda) = J + \int_0^{\lambda} A(u)\, \xi(u)\, du$$

where $A(\lambda)$ is the matrix $(a_{i,j}(\lambda))$ of the type $m \times m$: with the elements $a_{i,j}(\lambda) = s^{\beta_{i,j}} \omega_{i,j}(\lambda) \in \mathcal{C}^*(\lambda)$. $\xi(\lambda)$ is an element of the product $\prod_1^m \mathcal{C}^*(\lambda)$ in which the algebraical and topologycal structure are defined in the usual way.

The family of seminorms in $\prod_1^m \mathcal{C}^*(\lambda)$ is:

$$N_{p,q}(\xi) = \text{Max}\,(\| x_1(\lambda) \|_{p,q}, \; \| x_2(\lambda) \|_{p,q}, \ldots, \; \| x_m(\lambda) \|_{p,q}).$$

Let A^k be the product of k matrix

$$A^k = (a_{i,j}(\lambda_1))\,(a_{i,j}(\lambda_2)) \cdots (a_{i,j}(\lambda_k)).$$

The common element of this matrix is noted by

$$a_{i,j}^k(\lambda_1, \ldots, \lambda_k) = s^{\beta_{i,j}^k} \omega_{i,j}^k(\lambda_1, \ldots, \lambda_k)$$
$$= s^{\beta(k)}\, l^{\beta(k) - \beta_{i,j}^k} \omega_{i,j}^k(\lambda_1, \ldots, \lambda_k)$$

where $\beta(k) = \underset{1 \leq i,j \leq m}{\text{Max}} \beta_{i,j}^k$. The operator $l^{\beta(k)-\beta_{i,j}^k} \omega_{i,j}^k(\lambda_1, \ldots, \lambda_k)$ is defined by a continuous function on: $0 \leq \lambda_1, \lambda_2, \ldots, \lambda_k \leq \Lambda \quad 0 \leq t < \infty$.

In this case the operator T is:

$$T\xi = J + \int_0^\lambda A(u)\, \xi(u)\, du$$

and it maps $\prod_1^m M$ into itself.

In order to show that condition 1 of theorem 2 is satisfied, we shall give some inequalities. Let $M_q = \nu_k(\omega_{i,j}(\lambda))$, then:

$$|\omega_{i,j}^k(\lambda_1, \lambda_2, \ldots, \lambda_k, t)| \leq \frac{m^k M_q^k}{\Gamma(k)}$$

and

$$\left| l^{\beta(k)-\beta_{i,j}^k} \omega_{i,j}^k(\lambda_1, \ldots, \lambda_2) \right| \leq \frac{m^{k-1} M_q^k}{\Gamma(k)} \; \frac{t^{\beta(k)-\beta_{i,j}^k}}{\Gamma(\beta(k) - \beta_{i,j}^k + 1)}.$$

Now we can show for operator T related to equation (9)

$$N_{p,k}[T^k \xi - T^k \eta] = N_{p,q}[T_h^k(\xi - \eta)] \leq Q(k)\, N_{p+1,q}(\xi - \eta)$$

where

$$Q(k) = 0\left(k^{\left(\frac{\beta(k)}{\sigma} - 2\right)k}\right)$$

If $\beta(k) < 2$, the conditions of theorem 2 are satisfied. For the determination of $\beta(k)$ and for a consideration of the system (7) see [5] and [6].

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