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ON AN INEQUALITY OF N. LEVINSON

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N. Levinson [1] (see also [2]) has obtained the following result:

If f is three times differentiable on $(0, 2b)$ so that $f'''>0$ for $x\in(0, 2b)$ and if $x_i\in(0, b)$ $(i=1, \ldots, n)$, $p_i>0$ $(i=1, \ldots, n)$ then

$$(1)\quad \sum_{i=1}^{n} p_i f(x_i)-\sum_{i=1}^{n} p_i f(2b-x_i)\leqslant \sum_{i=1}^{n} p_i\left(f\left(\frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i}\right)-f\left(2b-\frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i}\right)\right).$$

If $f'''>0$ on $(0,2b)$ equality holds if and only if $x_1=\cdots=x_n$.

T. Popoviciu [3] has generalized the inequality (1) of N. Levinson giving other conditions under which it holds.

In this paper we shall give the lower bound for the expression

$$\sum_{i=1}^{n} p_i f(x_i)-\sum_{i=1}^{n} p_i f(2b-x_i).$$

T h e o r e m. *If the following conditions are fulfilled:*

1° *Inequality* (1) *holds for* $x_i\in[0, b]$;

2° $p_i\geqslant 1$ $(i=1, \ldots, n)$;

3° $\sum_{i=1}^{n} p_i x_i\in[0, b]$;

then we have

$$(2)\quad \sum_{i=1}^{n} p_i f(x_i)-\sum_{i=1}^{n} p_i f(2b-x_i)\geqslant f\left(\sum_{i=1}^{n} p_i x_i\right)-f\left(2b-\sum_{i=1}^{n} p_i x_i\right)$$
$$-\left(1-\sum_{i=1}^{n} p_i\right)(f(0)-f(2b)).$$

P r o o f. For $n=2$, (1) yields

$$pf(x)+qf(y)-pf(2b-x)-qf(2b-y)\leq \leq (p+q)\left[f\left(\frac{px+qy}{p+q}\right)-f\left(2b-\frac{px+qy}{p+q}\right)\right] \tag{3}$$

where $x, y\in[0, b]$ and $p, q>0$.

Putting in (3)

$$x=p_1x_1+p_2x_2, \quad y=0, \quad p=1, \quad q=p_1-1+p_2\frac{x_2}{x_1},$$

where $x_1, x_2\in[0, b]$ and $p_1x_1+p_2x_2\in[0, b]$, we get

$$f(p_1x_1+p_2x_2)+\left(p_1-1+p_2\frac{x_2}{x_1}\right)f(0)-f(2b-p_1x_1-p_2x_2)-\left(p_1-1+p_2\frac{x_2}{x_1}\right)f(2b)$$

$$\leq\left(p_1+p_2\frac{x_2}{x_1}\right)(f(x_1)-f(2b-x_1))$$

and, after multiplying by $\dfrac{p_1x_1}{p_1x_1+p_2x_2}$,

$$\frac{p_1x_1}{p_1x_1+p_2x_2}(f(p_1x_1+p_2x_2)-f(2b-p_1x_1-p_2x_2)) \leq p_1[f(x_1)-f(2b-x_1)]-\left(p_1-\frac{p_1x_1}{p_1x_1+p_2x_2}\right)[f(0)-f(2b)]. \tag{4}$$

If we permute in (4) p_1 and p_2 as well as x_1 and x_2, and add the obtained inequality to (4), we get

$$f(p_1x_1+p_2x_2)-f(2b-p_1x_1-p_2x_2)-(1-p_1-p_2)(f(0)-f(2b)) \leq p_1f(x_1)+p_2f(x_2)-p_1f(2b-x_1)-p_2f(2b-x_2). \tag{5}$$

Therefore, theorem 1 holds for $n=2$. Suppose now that theorem 1 holds for some n $(n\geq 2)$. Then, according to (5), we have

$$f\left(\sum_{i=1}^{n+1}p_ix_i\right)-f\left(2b-\sum_{i=1}^{n+1}p_ix_i\right)=f\left(\sum_{i=1}^{n}p_ix_i+p_{n+1}x_{n+1}\right)$$

$$-f\left(2b-\sum_{i=1}^{n}p_ix_i-p_{n+1}x_{n+1}\right)\leq f\left(\sum_{i=1}^{n}p_ix_i\right)+p_{n+1}f(x_{n+1})$$

$$-f\left(2b-\sum_{i=1}^{n}p_ix_i\right)-p_{n+1}f(2b-x_{n+1})$$

$$+(1-1-p_{n+1})(f(0)-f(2b)).$$

Using the inductive hypothesis, we obtain

$$f\left(\sum_{i=1}^{n+1} p_i x_i\right)-f\left(2b-\sum_{i=1}^{n+1} p_i x_i\right)$$

$$\leqslant \sum_{i=1}^{n} p_i f(x_i)-\sum_{i=1}^{n} p_i f(2b-x_i)+\left(1-\sum_{i=1}^{n} p_i\right)(f(0)-f(2b))$$

$$+p_{n+1} f(x_{n+1})-p_{n+1} f(2b-x_{n+1})-p_{n+1}(f(0)-f(2b))$$

$$=\sum_{i=1}^{n+1} p_i f(x_i)-\sum_{i=1}^{n+1} p_i f(2b-x_i+\left(1-\sum_{i=1}^{n+1} p_i\right)[f(0)-f(2b)].$$

Therefore, if (2) holds for n, it also holds for $n+1$, which completes the inductive proof.

From (1) and (2) for $p_i=1$ $(i=1, \ldots, n)$ we get the following result:

Theorem 2. *If f is three times differentiable on* $[0, 2b]$ *and* $f'''>0$ *for* $x \in [0, 2b]$ *and if* $x_i \in [0, b]$, $\sum_{i=1}^{n} x_i \in [0, b]$, *then*

$$f\left(\sum_{i=1}^{n} x_i\right)-f\left(2b-\sum_{i=1}^{n} x_i\right)-(1-n)(f(0)-f(2b)) \tag{6}$$

$$\leqslant \sum_{i=1}^{n} f(x_i)-\sum_{i=1}^{n} f(2b-x_i)$$

$$\leqslant n\left(f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)-f\left(2b-\frac{1}{n}\sum_{i=1}^{n} x_i\right)\right).$$

Remark. It can be seen from the proof that if $f'''>0$ for $x \in [0, 2b]$, than equality holds in (2) if and only if

$$x_1=\cdots=x_n=0. \tag{7}$$

Therefore, in the first inequality of (6) we have equality if and only if (7) holds, while in the second if and only if $x_1=\cdots=x_n$.

REFERENCES

[1] N. Levinson, *Generalisation of an inequality of Ky Fan*, J. Math. Anal. App. 8 (1964), 133—134.

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[3] T. Popoviciu, *Sur une inégalité de N. Levinson*, Mathematica (Cluj) 6 (29) (1964), 301—306.