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ON UNSTEADY COMPRESSIBLE LAMINAR BOUNDARY LAYERS PAST BODIES OF REVOLUTION SPINNING ABOUT ITS AXIS

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1. Introduction

In recent years, theoretical studies on boundary layer growth past an axially symmetric body have gained considerable interest due to its numerous applications to problems of Engineering. Namely, Boltze [1], Ašković [2], [3] and Warsi [4] have considered the case of boundary layer growth on axially symmetric bodies when the velocity of the stream is proportional with degree or exponential change of time. Further, Illingworth [5], Wadhwa [6] and Đurić [7], [8] have solved the problem of unsteady laminar boundary layer on axially symmetric body wich is put to spiral motion. In all these cases the fluid is considered to be everywhere incompressible.

Recently Brown [9] has analyzed the effect of heat transfer on two dimensional boundary layer growth while more recently Riley [10] has investigated boundary layer flows which are induced when an isothermal rigid-body rotation is disturbed by heating the fluid.

This paper is dedicated to the question of solving unsteady compressible boundary layer in the impulsive motions of an axisymmetric body which executes a translatory motion in the direction of his axis of symmetry and at the same time rotates about it. We confine our attention to fluids of small viscosity and Prandtl number equal to unity. Such flows will be described by the boundary layer equations which may be further simplified if we assume a linear variation of viscosity with temperature. The assumption is made that the effects of compressibility are confined to the boundary layer and the main stream remains incompressible. This could be realized in practice by releasing a stream of small Mach number past a very hot body with time-independent velocity.

2. Equations of motion

For fluids of small viscosity μ the compressible boundary layer equations (momentum, energy, mass conservation and state) which govern unsteady flow past a body of revolution spinning about its axis, which is parallel to the stream are

$$(1)\quad \begin{cases} \rho\left(\dfrac{\partial u}{\partial t}+u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y}-\dfrac{w^2}{r}\dfrac{dr}{dx}\right)=-\dfrac{\partial p}{\partial x}+\dfrac{\partial}{\partial y}\left(\mu\dfrac{\partial u}{\partial y}\right), \\ \rho\left(\dfrac{\partial w}{\partial t}+u\dfrac{\partial w}{\partial x}+v\dfrac{\partial w}{\partial y}+\dfrac{uw}{r}\dfrac{\partial r}{\partial x}\right)=\dfrac{\partial}{\partial y}\left(\mu\dfrac{\partial w}{\partial y}\right), \end{cases}$$

$$(2)\quad \rho\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=\frac{1}{\sigma}\frac{\partial}{\partial y}\left(\mu\frac{\partial T}{\partial y}\right),$$

$$(3)\quad \frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial x}(\rho r u)+\frac{\partial}{\partial y}(\rho v)=0,$$

$$(4)\quad p=\rho RT.$$

The boundary conditions being

$$(5)\quad \begin{aligned} &u=v=0, \quad w=r\Omega, \quad T=T_w \quad \text{for} \quad y=0 \quad \text{and } t>0, \\ &u=U(x), \quad w=0, \quad T=T_\infty \quad \text{for} \quad y=\infty. \end{aligned}$$

In these equations x and y are measured along and normal to the wall and z is the distance along the arc of a transversal crossection; $u(x, y. t)$, $v(x, y, t)$ and $w(x, y, t)$ are the components of velocity in the direction of x, y and z respectively; t represents time; $r(x)$ radius of transversal crossection; p denotes pressure; T temperature; σ Prandtl number; $U(x)$ main-stream; Ω angular velocity; T_w and T_∞ are the constant wall and ambient temperature respectively. The density ρ and viscosity μ are functions of T and we shall assume that the fluid is a gas such that

$$(6)\quad \mu \sim T.$$

In the energy equation (2) terms representing viscous dissipation have been neglected, thus $|T_w-T_\infty| \gg L^2\Omega^2/c_p$ where L is a typical length and c_p the coefficient of specific heat at constant pressure.

With the assumption (6) it is possible to transform the above equations as follows. We introduce a coordinate Y defined by

$$(7)\quad Y=\int_0^y \frac{\rho}{\rho_\infty}\,dy,$$

and a stream function Ψ such that

$$(8)\quad u=\frac{1}{r}\frac{\rho_\infty}{\rho}\frac{\partial\Psi}{\partial y}=\frac{1}{r}\frac{\partial\Psi}{\partial Y}.$$

From the continuity equation (3) we then have

$$(9)\quad \frac{\rho}{\rho_\infty}v=-\left\{\frac{\partial Y}{\partial t}+\frac{1}{r}\frac{\partial\Psi}{\partial x}+\frac{1}{r}\frac{\partial\Psi}{\partial Y}\left(\frac{\partial Y}{\partial x}\right)_{y,\,t}\right\}.$$

Thus, using (4), (6) (7), (8) and (9) together with $\rho\mu=\text{const}=\rho_\infty\mu_\infty$, a consequence of (5) and (6), equations (1) and (3) may be written as

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{1}{r}\frac{\partial \Psi}{\partial Y}\right)+\frac{1}{r}\frac{\partial \Psi}{\partial Y}\frac{\partial}{\partial x}\left(\frac{1}{r}\frac{\partial \Psi}{\partial Y}\right)-\frac{1}{r}\frac{\partial \Psi}{\partial x}\frac{\partial}{\partial Y}\left(\frac{1}{r}\frac{\partial \Psi}{\partial Y}\right)-\\
&\qquad\qquad\frac{w^2}{r}\frac{dr}{dx}=U\frac{dU}{dx}(1+S)+\nu_\infty\frac{\partial^2}{\partial Y^2}\left(\frac{1}{r}\frac{\partial \Psi}{\partial Y}\right),\\
&\frac{\partial w}{\partial t}+\frac{1}{r}\frac{\partial \Psi}{\partial Y}\frac{\partial w}{\partial x}-\frac{1}{r}\frac{\partial \Psi}{\partial x}\frac{\partial w}{\partial Y}+\frac{w}{r^2}\frac{\partial \Psi}{\partial Y}\frac{dr}{dx}=\nu_\infty\frac{\partial^2 w}{\partial Y^2},\\
&\frac{\partial S}{\partial t}+\frac{1}{r}\frac{\partial \Psi}{\partial Y}\frac{\partial S}{\partial x}-\frac{1}{r}\frac{\partial \Psi}{\partial x}\frac{\partial S}{\partial Y}=\frac{\nu_\infty}{\sigma}\frac{\partial^2 S}{\partial Y^2},
\end{aligned}
\tag{10}
$$

where $\nu=\mu/\rho$ is the kinematic viscosity and $T/T_\infty=1+S$.

2. Solution of the problem

For solving this problem the method of succesive approximations is applied which has its physical meaning in connection with the process of forming the boundary layer. Thus if we assume that Ψ and w have the forms

$$
\begin{aligned}
&\Psi(x, Y, t)=2\sqrt{\nu_\infty t}\, rUF(x, \eta, t),\\
&w(x, Y, t)=\Omega\, r\, \Phi(x. \eta, t),
\end{aligned}
\tag{11}
$$

where $\eta=Y/2\sqrt{\nu_\infty t}$ is the similarity variable, the equations (10) become

$$
\begin{aligned}
&\frac{\partial^3 F}{\partial\eta^3}+2\eta\frac{\partial^2 F}{\partial\eta^2}-4t\frac{\partial^2 F}{\partial t\,\partial\eta}+4t\left\{\frac{dU}{dx}\left[1+S-\left(\frac{\partial F}{\partial\eta}\right)^2+F\frac{\partial^2 F}{\partial\eta^2}\right]+\right.\\
&\quad\left.U\left(\frac{\partial F}{\partial x}\frac{\partial^2 F}{\partial\eta^2}-\frac{\partial F}{\partial\eta}\frac{\partial^2 F}{\partial x\,\partial\eta}\right)+\frac{UF}{r}\frac{dr}{dx}\frac{\partial^2 F}{\partial\eta^2}+\Omega^2\frac{r}{U}\frac{dr}{dx}\Phi^2\right\}=0,\\
&\frac{\partial^2\Phi}{\partial\eta^2}+2\eta\frac{\partial\Phi}{\partial\eta}-4t\frac{\partial\Phi}{\partial t}+4t\left[F\frac{dU}{dx}\frac{\partial\Phi}{\partial\eta}+\right.\\
&\quad\left.U\left(\frac{\partial F}{\partial x}\frac{\partial\Phi}{\partial\eta}-\frac{\partial F}{\partial\eta}\frac{\partial\Phi}{\partial x}\right)+\frac{U}{r}\frac{dr}{dx}\left(F\frac{\partial\Phi}{\partial\eta}-2\Phi\frac{\partial F}{\partial\eta}\right)\right]=0,\\
&\frac{1}{\sigma}\frac{\partial^2 S}{\partial\eta^2}+2\eta\frac{\partial S}{\partial\eta}-4t\frac{\partial S}{\partial t}+4t\left[\frac{F}{r}\frac{d}{dx}(UR)\frac{\partial S}{\partial\eta}+\right.\\
&\qquad\qquad\left.U\left(\frac{\partial F}{\partial x}\frac{\partial S}{\partial\eta}-\frac{\partial F}{\partial\eta}\frac{\partial S}{\partial x}\right)\right]=0,
\end{aligned}
\tag{12}
$$

subjected to the boundary conditions

$$
\begin{aligned}
&F=\frac{\partial F}{\partial\eta}=0,\quad \Phi=1,\quad S=T_w/T_\infty-1\equiv S_w(\text{const})\quad \text{for } \eta=0,\\
&\frac{\partial F}{\partial\eta}=1,\qquad \Phi=0,\quad S=0\qquad\qquad \text{for } \eta=\infty.
\end{aligned}
\tag{13}
$$

Now the equations (12) show that solutions for the fuctions F, and S may be found in the forms

$$F = F_0(\eta) + t\left[\frac{dU}{dx}F_{11}(\eta) + \frac{U}{r}\frac{dr}{dx}F_{12}(\eta) + \Omega^2\frac{r}{U}\frac{dr}{dx}F_{13}(\eta)\right] + ---,$$

$$\Phi = \Phi_0(\eta) + t\left[\frac{dU}{dx}\Phi_{11}(\eta) + \frac{U}{r}\frac{dr}{dx}\Phi_{12}(\eta)\right] + ---, \tag{14}$$

$$S = S_0(\eta) + t\frac{1}{r}\frac{d}{dx}(Ur)S_{11}(\eta) + ---.$$

Substituting these assumed solutions into the equations (12) each equation will separate itself into the system of ordinary differential equations for determination of coefficients-functions of every of above solutions. Therefore we have the following recursive system of the ordinary differential equations

$$\begin{gathered} F_0''' + 2\eta F_0'' = 0, \quad \Phi_0'' + 2\eta\Phi_0' = 0, \quad S_0'' + 2\eta S_0' = 0; \\ F_{11}''' + 2\eta F_{11}'' - 4F_{11}' = 4(-1 + F_0'^2 - F_0 F_0'') - 4S_0, \\ \Phi_{11}'' + 2\eta\Phi_{11}' - 4\Phi_{11} = -4F_0\Phi_0', \\ S_{11}'' + 2\eta S_{11}' - 4S_{11} = -4F_0 S_0'; \\ F_{12}''' + 2\eta F_{12}'' - 4F_{12}' = -4F_0F_0'', \\ \Phi_{12}'' + 2\eta\Phi_{12}' - 4\Phi_{12} = -4(F_0\Phi_0' - 2\Phi_0 F_0'), \\ F_{13}''' + 2\eta F_{13}'' - 4F_{13}' = -4\Phi_0^2, \end{gathered} \tag{15}$$

where primes denote the differentiation with respect to η. The boundary conditions (13) transform themselves into the following forms

$$\begin{aligned} &\left.\begin{aligned} &F_0 = F_0' = F_{11} = F_{11}' = F_{12} = F_{12}' = F_{13} = F_{13}' = 0, \\ &\Phi_0 = 1, \; S_0 = S_w, \; \Phi_{11} = \Phi_{12} = S_{11} = 0 \end{aligned}\right\} \text{ at } \eta = 0, \\ &\left.\begin{aligned} &F_0' = 1, \; F_{11}' = F_{12}' = F_{13}' = 0, \\ &\Phi_0 = S_0 = \Phi_{11} = S_{11} = \Phi_{12} = 0 \end{aligned}\right\} \text{ at } \eta = \infty. \end{aligned} \tag{16}$$

The solutions of first three equations belonging to the recursive system (15) are

$$F_0(\eta) = \eta\,\mathrm{erf}\,\eta + \frac{1}{\sqrt{\pi}}(e^{-\eta^2} - 1), \quad \Phi_0(\eta) = S_0(\eta)/S_w = 1 - \mathrm{erf}\,\eta, \tag{17}$$

where

$$\mathrm{erf}\,\eta = \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-\gamma^2}\,d\gamma,$$

is the Gauss' error function.

Now, with (17), the second equation of (15) becomes

$$F_{11}''' + 2\eta F_{11}'' - 4F_{11}' = 4\,\mathrm{erf}^2\eta - 4\left(\frac{2}{\sqrt{\pi}}\eta e^{-\eta^2} - S_w\right)\mathrm{erf}\,\eta - \frac{8}{\pi}e^{-2\eta^2} + \frac{8}{\pi}e^{-\eta^2} - 4(S_w + 1). \tag{18}$$

The particular solution of the non-homogeneous part of the above equation is to be found in the form

$$F_{11p}'(\eta) = X_{11}(\eta)\,\mathrm{erf}^2\eta + Y_{11}(\eta)\,\mathrm{erf}\,\eta + Z_{11}(\eta). \tag{19}$$

The substitution of (19) into (18) for the unknow coefficients-functions $X_{11}(\eta)$, $Y_{11}(\eta)$ and $Z_{11}(\eta)$, leads to the differential equations

$$X_{11}'' + 2\eta X_{11}' - 4X_{11} = 4,$$

$$Y_{11}'' + 2\eta Y_{11}' - 4Y_{11} = -\frac{8}{\sqrt{\pi}}X_{11}'\cdot e^{-\eta^2} - \frac{8}{\sqrt{\pi}}\eta e^{-\eta^2} + 4S_w,$$

$$Z_{11}'' + 2\eta Z_{11}' - 4Z_1^{\mathrm{I}} = -\frac{4}{\sqrt{\pi}}Y_{11}'\cdot e^{-\eta^2} - \frac{8}{\pi}X_{11}\cdot e^{-2\eta^2} + \frac{8}{\pi}e^{\eta^2} - \frac{8}{\pi}e^{-2\eta^2} - 4(S_w + 1).$$

Solving this system we obtain

$$X_{11}(\eta) = \eta^2 + \frac{1}{2},\quad Y_{11}(\eta) = \frac{3}{\sqrt{\pi}}\eta e^{-\eta^2} - S_w,$$

$$Z_{11}(\eta) = \frac{2}{\pi}e^{-2\eta^2} - \frac{4}{3\pi}e^{-\eta^2} + S_w + 1.$$

Hence, the particular integral (19) becomes completely definite. Since the particular solution of homogeneus part of the differential equation (18) is

$$F_{11h}'(\eta) = C_1(1 + 2\eta^2) + C^2\left[(1 + 2\eta^2)\,\mathrm{erf}\,\eta + \frac{2}{\sqrt{\pi}}\eta e^{-\eta^2}\right],$$

it is possible to formulate the general solution of the initial equation

$$F_{11}'(\eta) = C_1(1 + 2\eta^2) + C_2\left[(1 + 2\eta^2)\,\mathrm{erf}\,\eta + \frac{2}{\sqrt{\pi}}\eta e^{-\eta^2}\right] + F_{11p}'(\eta).$$

With boundary conditions (16) we obtain the following values for the constants

$$C_1 = -\left(1 + \frac{2}{3\pi} - S_w\right),\quad C_2 = \frac{1}{2} + \frac{2}{3\pi} + S_w.$$

Finally, solving in such way other equations (15) we are able to write finite forms of solutions of systems of equations (15) as follows

$$F'_{11}(\eta) = -\left(1 + \frac{2}{3\pi} + S_w\right)(1 + 2\eta^2) + \left(\frac{1}{2} + \frac{2}{3\pi} + S_w\right)\left[(1 + 2\eta^2)\operatorname{erf}\eta + \frac{2}{\sqrt{\pi}}\eta e^{-\eta^2}\right] + \left(\eta^2 - \frac{1}{2}\right)\operatorname{erf}^2\eta + \frac{3}{\sqrt{\pi}}\eta e^{-\eta^2}\operatorname{erf}\eta + \frac{2}{\pi}e^{-2\eta^2} - \frac{4}{3\pi}e^{-\eta^2} + S_w(1 - \operatorname{erf}\eta) - 1,$$

$$\Phi_{11}(\eta) = S_{11}(\eta)/S_w = \left(\frac{4}{3\pi} - \frac{1}{2}\right)\left[(1 + 2\eta^2)\operatorname{erf}\eta + \frac{2}{\sqrt{\pi}}\eta e^{-\eta^2}\right] + \left(\frac{1}{2} + \eta^2\right)\operatorname{erf}^2\eta + \frac{1}{\sqrt{\pi}}\eta e^{-\eta^2}\operatorname{erf}\eta + \frac{4}{3\pi}e^{-\eta^2} - \frac{4}{3\pi}(1 + 2\eta^2),$$

$$F'_{12}(\eta) = \frac{4}{3\pi}(1 + 2\eta^2) + \left(\frac{1}{2} - \frac{4}{3\pi}\right)\left[(1 + 2\eta^2)\operatorname{erf}\eta + \frac{2}{\sqrt{\pi}}\eta e^{-\eta^2}\right] - \left(\frac{1}{2} + \eta^2\right)\operatorname{erf}^2\eta - \frac{1}{\sqrt{\pi}}\eta e^{-\eta^2}\operatorname{erf}\eta - \frac{4}{3\pi}e^{-\eta^2},$$

$$F'_{13}(\eta) = \left(\frac{2}{\pi} - 1\right)(1 + 2\eta^2) + 2\left(1 - \frac{1}{\pi}\right)\left[(1 + 2\eta^2)\operatorname{erf}\eta + \frac{2}{\sqrt{\pi}}\eta e^{-\eta^2}\right] - 2\eta^2\operatorname{erf}^2\eta - 2\left(1 + \frac{2}{\sqrt{\pi}}\eta e^{-\eta^2}\right)\operatorname{erf}\eta - \frac{2}{\pi}e^{2\eta^2} + 1.$$

The position of separation of forward flow from the contour is given by $\partial u/\partial y|_{y=0} = 0$ (or $\partial u/\partial \eta|_{\eta=0} = 0$) and the time t_s at which separation occurs at any particular place is therefore given by

$$t_s = -\frac{F''_0(0)}{\dfrac{dU}{dx}F''_{11}(0) + \dfrac{U}{r}\dfrac{dr}{dx}F''_{12}(0) + \Omega^2\dfrac{r}{U}\dfrac{dr}{dx}F''_{13}(0)}$$

where

$$F''_{12}(0) = \frac{2}{\sqrt{\pi}}, \quad F''_{11}(0) = \frac{2}{\sqrt{\pi}}\left(1 + \frac{4}{3\pi} + S_w\right),$$

(20)

$$F''_{12}(0) = \frac{2}{\sqrt{\pi}}\left(1 - \frac{8}{3\pi}\right), \quad F''_{13}(0) = \frac{4}{\sqrt{\pi}}\left(1 - \frac{2}{\pi}\right).$$

4. Application

Let the sphere of radius a put into spiral motion by an impulsively jerk. In this case we have

$$r(x) = a\sin x/a, \quad U(x) = \frac{3}{2}U_\infty \sin x/a,$$

where U_∞ is the constant velocity of the fluid relative to the sphere. Therefore, as a first approximation, the time of separation is given by

(21) $$t_s^* = -\frac{F_0''(0)}{\cos x/a\left[3/2\left(F_{11}''(0)+F_{12}''(0)\right)+\frac{2}{3}\overline{\Omega}F_{13}''(0)\right]}$$

where

$$t_s^* = U_\infty t_s/a, \qquad \overline{\Omega} = (\Omega a/U_\infty)^2.$$

Hence as in the incompressible case separation corresponds to having $\cos x/a = -1$ i.e. the last stagnation point. Using (20) relation (21) becomes

(22) $$t_s^* = \frac{1}{2.3634 + 1.5\,S_w + 0.4845\,\overline{\Omega}}$$

from where it follows that in the heat transfer case separation occurs earlier than in the incompressible case.

When the wall temperature is large keeping $\overline{\Omega}$ fixed then relation (22) shows that

$$t_s^* \sim 1/S_w \quad \text{as} \quad S_w \to \infty,$$

so that for very large wall temperatures separation occurs almost instantaneously.

The values of t_s^* for different values of S_w and $\overline{\Omega}$ calculated from the formula (22) are given in the table below.

Now, in order to study the third approximations in t from (14) it is necessary to solve a system of linear differential equations of the same type as those of the second approximations. But, because the solutions are very complicated we will not occupy ourselves with these in this paper.

Table t_s^*

$\overline{\Omega}$	S_w		
	0	0.5	1
0	0.423	0.321	0.259
0.1	0.415	0.316	0.255
1	0.315	0.217	0.187
10	0.139	0.125	0.115

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