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COMPLETE INTEGRALS OF A CLASS OF LINEAR HOMOGENEOUS PARTIAL DIFFERENTIAL EQUATIONS

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1. M. Arsenović [1] has given a method for determining complete integrals of linear partial differential equations with constant coefficients of second or higher order. For example, the complete integral of the equation

$$Au_{xx}+2Bu_{xy}+Cu_{yy}+2Du_x+Eu_y+Fu=0,$$

where A, B, C, D, E, F are constants with $A\neq 0$, is given by

$$u=C_2 e^{\lambda_1(x-C_1 y)}+C_3 e^{\lambda_2(x-C_1 y)}+C_4 e^{\mu_1 x}+C_5 e^{\mu_2 x},$$

where λ_1, λ_2, μ_1, μ_2 are roots of the following equations

$$(CC_1^2-2BC_1+A)\lambda^2+2(D-C_1 E)\lambda+F=0,$$

$$A\mu^2+2D\mu+F=0,$$

respectively, and where C_1, C_2, C_3, C_4, C_5 are arbitrary constants.

In this article we shall use a very elementary method to obtain complete integrals of a class of linear partial differential equations with variable coefficients.

2. Consider the equation

$$a(x)\beta(y)u_{xx}+b(x,y)u_{xy}+\alpha(x)c(y)u_{yy}$$
$$+d(x)\beta(y)u_x+\alpha(x)e(y)u_y+K\alpha(x)\beta(y)u=0, \tag{1}$$

where K is a constant.

We shall look for a solution of (1) which is only a function of x, i.e., $u=F(x)$. We find

$$a(x)F''(x)+d(x)F'(x)+K\alpha(x)F(x)=0. \tag{2}$$

Suppose that

$$F(x)=\overline{C}_1 f_1(x)+C_2 f_2(x) \tag{3}$$

is the general solution of (2). Then, (3) is a solution of (1).

Similarly, if

$$G(y)=C_3 g_1(y)+C_4 g_2(y) \tag{4}$$

is the general solution of differential equation

$$c(y)\,G''(y)+e(y)\,G'(y)+K\beta(y)\,G(y)=0,$$

then (4) is also a solution of (1).

Therefore,

$$u(x, y)=\overline{C}_1 f_1(x)+C_2 f_2(x)+C_3 g_1(y)+C_4 g_2(y) \tag{5}$$

is a solution of (1).

We shall now look for a solution of (1) which is of the form

$$u(x, y)=\xi(x)\,\eta(y).$$

Differentiating and substituting this solution into (1), we find

$$\begin{aligned} &\beta(y)\,\eta(y)\,[a(x)\,\xi''(x)+d(x)\,\xi'(x)+K\alpha(x)\,\xi(x)] \\ &\quad+\alpha(x)\,c(y)\,\xi(x)\,\eta''(y)+[b(x, y)\,\xi'(x)+\alpha(x)\,e(y)\,\xi(x)]\,\eta'(y)=0. \end{aligned} \tag{6}$$

It is, therefore, convenient to choose $\xi(x)=f_1(x)$. Then (6) becomes

$$\eta''(y)+\left(\frac{b(x, y) f_1'(x)}{\alpha(x)\,c(y) f_1(x)}+\frac{e(y)}{c(y)}\right)\eta'(y)=0. \tag{7}$$

Thus, if

$$\frac{\partial}{\partial x}\left(\frac{b(x, y) f_1'(x)}{\alpha(x) c(y) f_1(x)}\right)=0,$$

we may integrate equation (7). Its solution will be of the form

$$\eta=C_5 h(y)+C_6,$$

which means that

$$u=C_5 f_1(x)\,h(y)+C_6 f_1(x) \tag{8}$$

is a solution of (1).

Combining (5) and (8), we obtain the following solution of (1):

$$u(x, y)=C_1 f_1(x)+C_2 f_2(x)+C_3 g_1(y)+C_4 g_2(y)+C_5 f_1(x)\,h(y). \tag{9}$$

Solution (9) contains 5 arbitrary constants. Moreover, it is linear with respect to the constants, whose elimination, therefore, leads only to (1). That means that (9) is the complete integral of (1).

Remark. (6) can be written in the form

$$\begin{aligned} &\alpha(x)\,\xi(x)\,[c(y)\,\eta''(y)+e(y)\,\eta'(y)+K\beta(y)\,\eta(y)] \\ &\quad+a(x)\,\beta(y)\,\eta(y)\,\xi''(x)+[b(x, y)\,\eta'(y)+d(x)\,\beta(y)\,\eta(y)]\,\xi'(x)=0. \end{aligned}$$

Therefore, if we put $\eta(y)=g_1(y)$, and if

$$\frac{\partial}{\partial y}\left(\frac{b(x,\ y)\,g_1'(y)}{a(x)\,\beta(y)\,g_1(y)}\right)=0,$$

we may use the same procedure to obtain the complete integral of (1).

3. Example.

The complete integral of

$$Au_{xx}+2Bu_{xy}+Cu_{yy}+2Du_x+2Eu_y+Fu=0,$$

where A, B, C, D, E, F are constants, is

$$u(x,\ y)=C_1\exp(\lambda_1 x)+C_2\exp(\lambda_2 x)+C_3\exp(\mu_1 y) + C_4\exp(\mu_2 y)+C_5\exp\left(\lambda_1 x-\frac{2B\lambda_1+2E}{C}\,y\right), \tag{10}$$

where λ_1, λ_2 $(\lambda_1\neq\lambda_2)$ are roots of the equation

$$A\lambda^2+2D\lambda+F=0,$$

and μ_1, μ_2 $(\mu_1\neq\mu_2)$ are roots of the equation

$$C\mu^2+2E\mu+F=0.$$

If, for example $\lambda_1=\lambda_2=\lambda$, then the expression $C_1\exp(\lambda_1 x)+C_2\exp(\lambda_2 x)$ in (10) should be replaced by $(C_1+C_2 x)\,e^{\lambda x}$.

Notice that the complete integral (10) is in a way more convenient than the one obtained in [1], as it is linear with respect to the arbitrary constants C_1, C_2, C_3, C_4, C_5.

REFERENCE

[1] M. Arsenović, *Sur l'intégration d'équations aux dérivées partielles du second ordre et d'ordre supérieur.* Bull. Soc. math. phys. Serbie 9 (1957), 193-196.